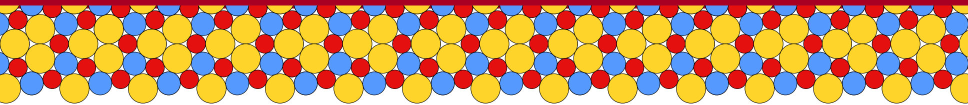
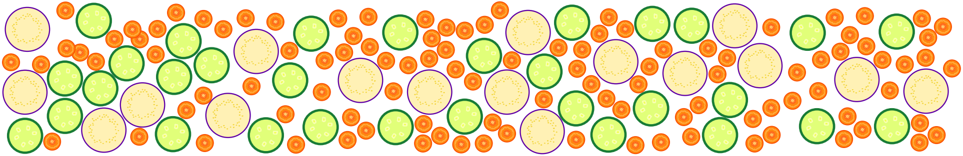


Triangulated ternary disc packings that maximize the density

Daria Pchelina

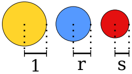
supervised by

Thomas Fernique

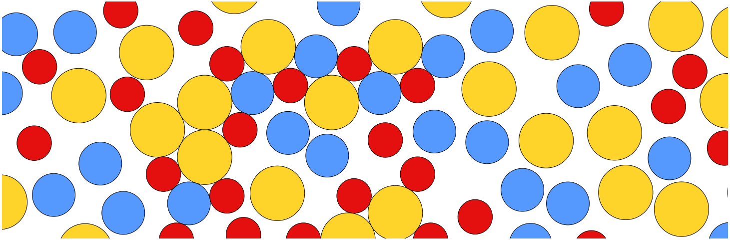


December 3, 2020

Discs:

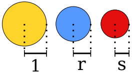


Packing P:
(in R^2)



What is a packing?

Discs:



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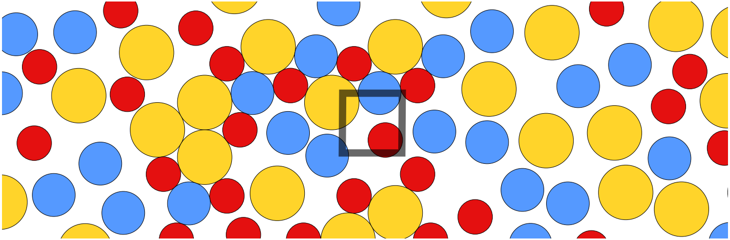
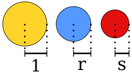


Density:

$$\delta(P) = \limsup_{n\to\infty} \frac{area([-n,n]^2 \cap P)}{area([-n,n]^2)}$$

Discs:



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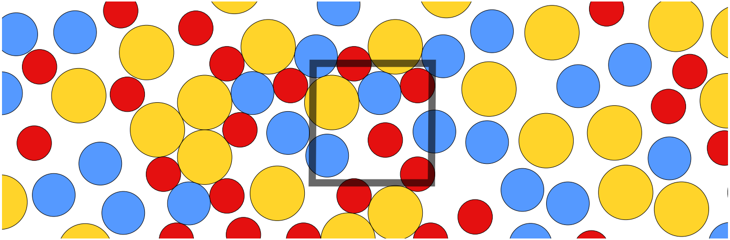
(in R^2)



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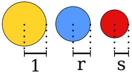
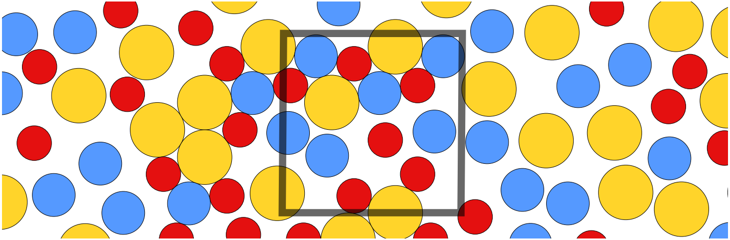


Packing P:
(in R^2)



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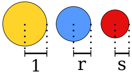

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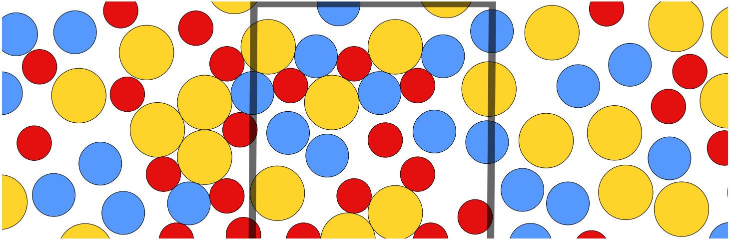

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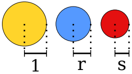

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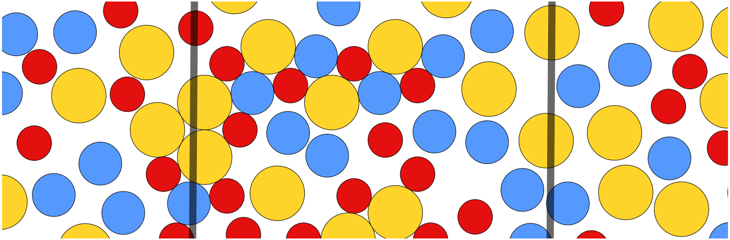

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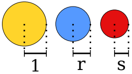
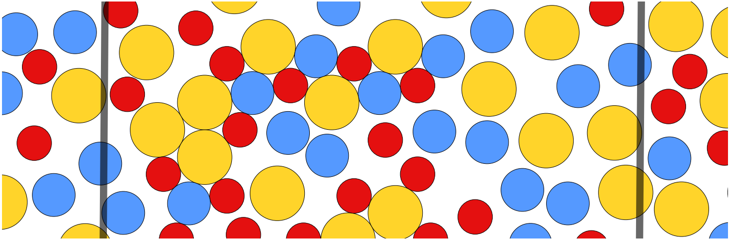
What is a packing?

Discs:



Packing P:
(in R^2)



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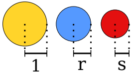
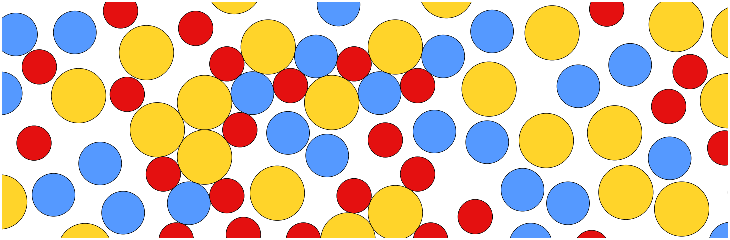


Packing P:
(in R^2)



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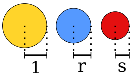
Which packings maximize the density?

Discs:



Packing P:
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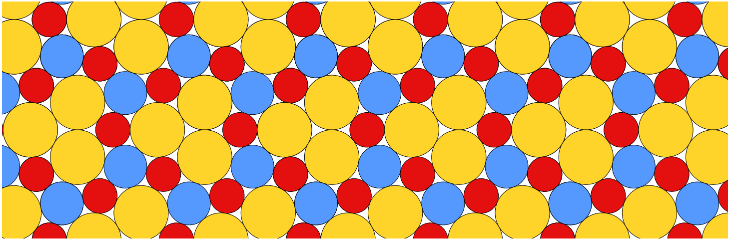

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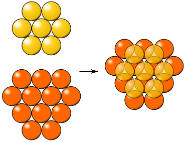
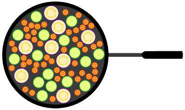
Which packings maximize the density?

- to pack fruits

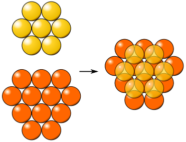


- and vegetables

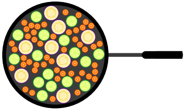


Why do we study packings?

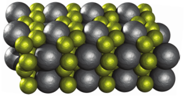
- to pack fruits

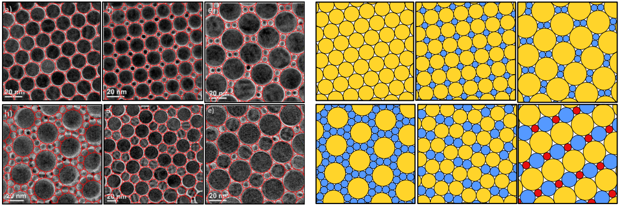


- and vegetables

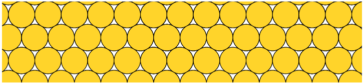


- to make compact materials





Binary and ternary superlattices self-assembled from colloidal nanodisks and nanorods. Journal of the American Chemical Society, 137(20):6662–6669, 2015.

2D hexagonal -packing:  $\delta = \frac{\pi}{2\sqrt{3}}$

Lagrange, 1772

The hexagonal packing maximizes the density among -lattice packings.

Thue, 1910 (Toth, 1940)

The hexagonal packing maximizes the density.

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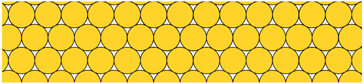

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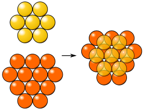
3D hexagonal -packing:



$\delta = \frac{\pi}{3\sqrt{2}}$

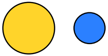
Gauss, 1831

The hexagonal packing maximizes the density among lattice -packings.

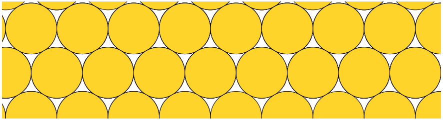
Hales, Ferguson, 1998–2014 (Conjectured by Kepler, 1611)

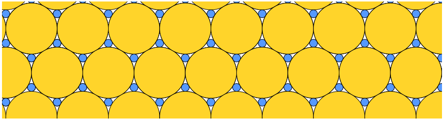
The hexagonal packing maximizes the density.

Two discs of radii 1 and r:



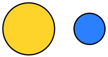
Lower bound on the density: $\frac{\pi}{2\sqrt{3}}$ (hexagonal packing with only 1 disc used)





Context

Two discs of radii 1 and r:



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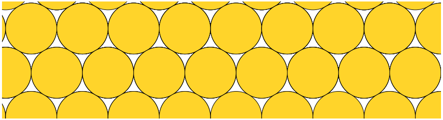
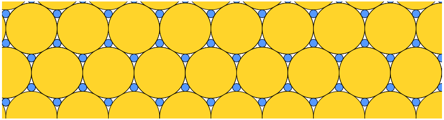
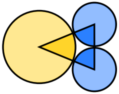




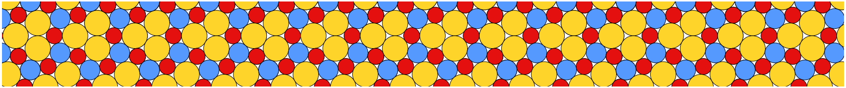
Upper bound on the density:

Florian, 1960

The density of a packing never exceeds the density in the following triangle:

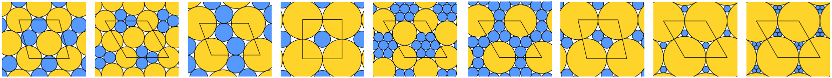


A packing is called **triangulated** if each “hole” is bounded by three tangent discs.



Kennedy, 2006

There are 9 values of r allowing triangulated packings.



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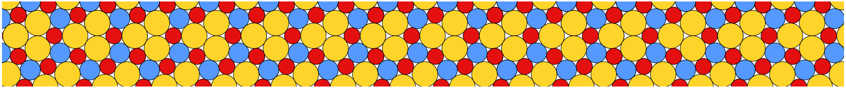

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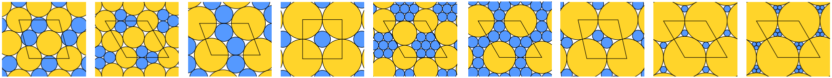
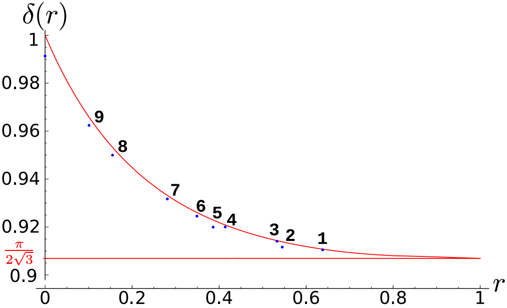


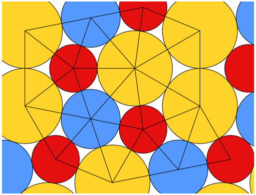
Heppes 2000,2003
Kennedy 2004
Bedaride, Fernique, 2019

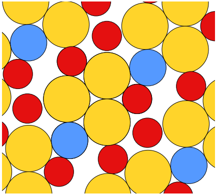
All these 9 packings maximize the density.

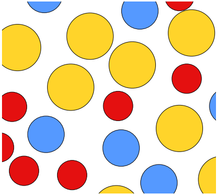


Conjecture (Connelly, 2018)

If a finite set of discs allows a **saturated** triangulated packing then the density is maximized on a saturated triangulated packing.







True for and .

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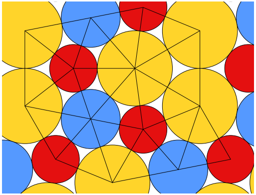
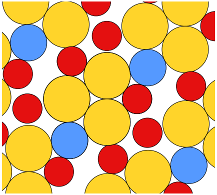
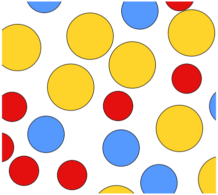


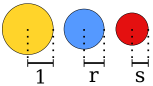




True for and .

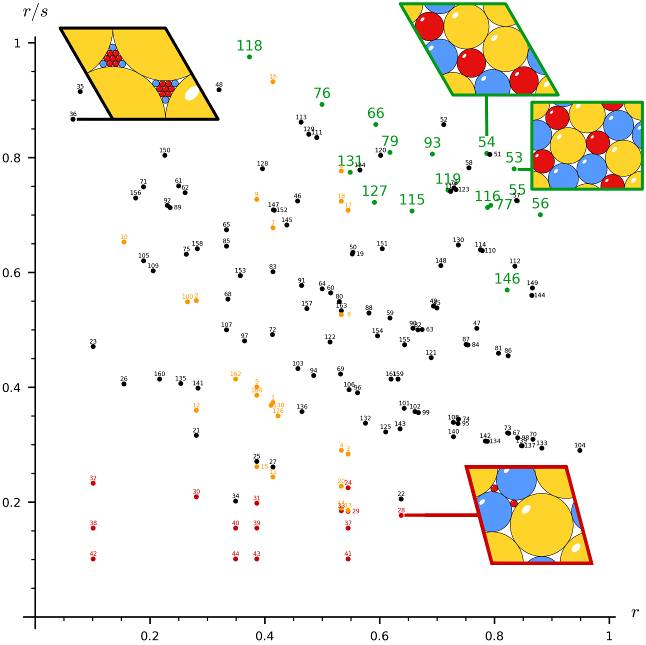
What happens with ?

3 discs:



164 pairs (r, s) allowing triangulated packings:

(Fernique, Hashemi, Sizova 2019)



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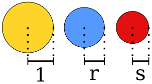

164 pairs (r, s) allowing triangulated packings:

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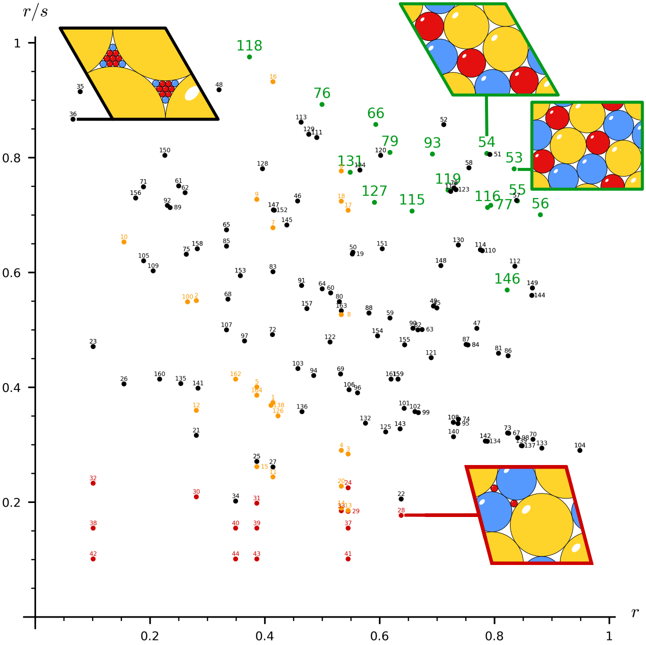
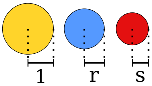
- 15 cases: non saturated
- 24 cases: a 2-disk packing is denser
- Case 53 is proved

  (Fernique 2019)



3 discs: 

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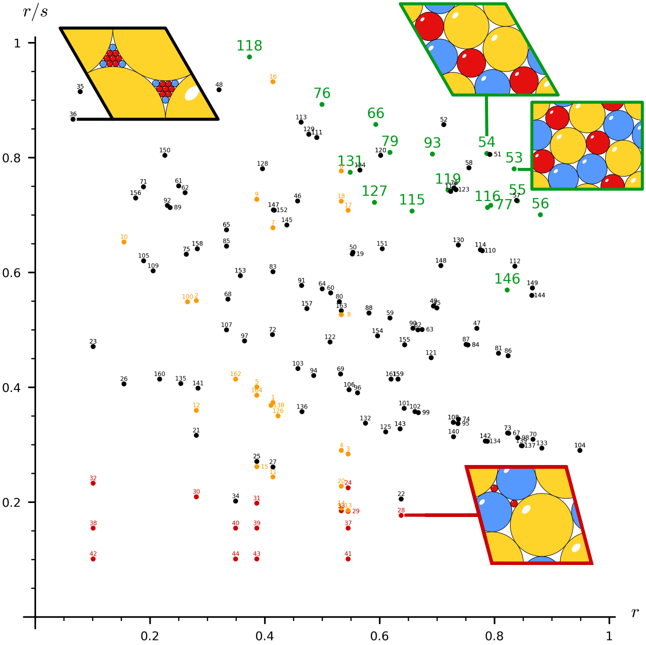
- **15 more cases proved**



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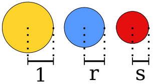

164 pairs (r, s) allowing triangulated packings:

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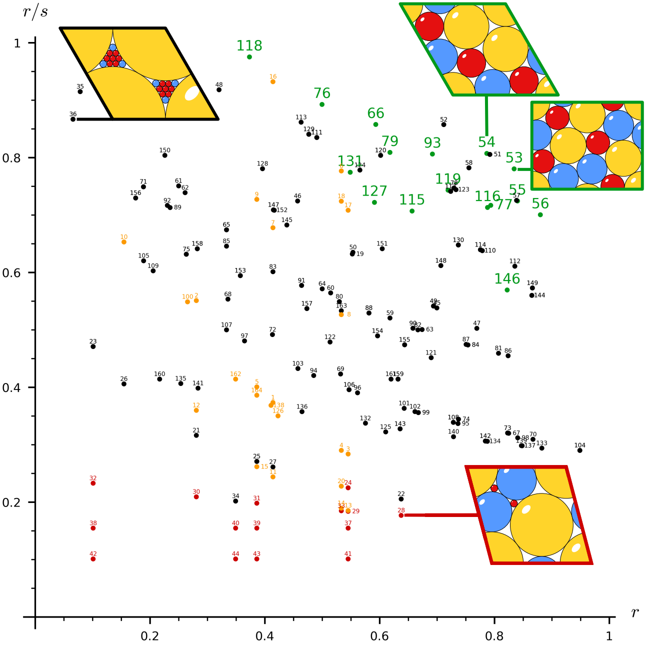
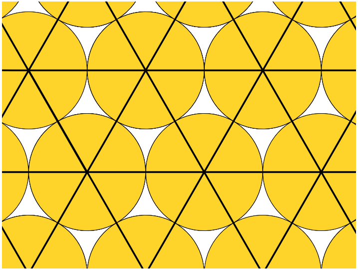
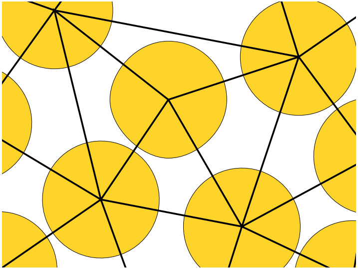
  (Fernique 2019)
- **15 more cases proved**
- The others?



A **Delaunay triangulation** of a packing: no points inside a circumscribed circle

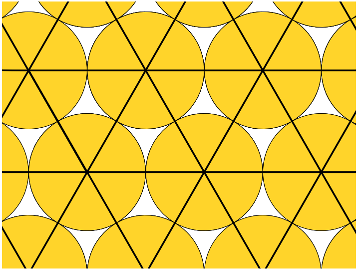


$$\delta^* = \delta_{\triangle^*} = \frac{\pi}{2\sqrt{3}}$$



$$\forall \triangle,\ \delta_{\triangle} \leq \delta_{\triangle^*} = \delta^*$$

A **Delaunay triangulation** of a packing: no points inside a circumscribed circle



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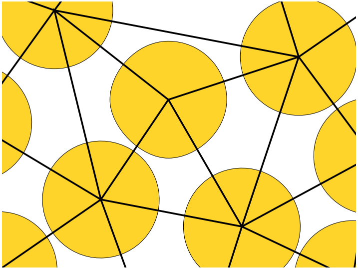

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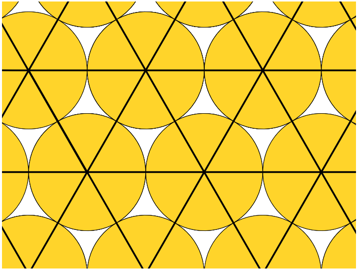

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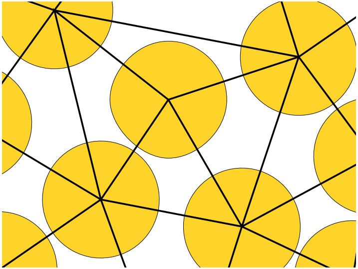
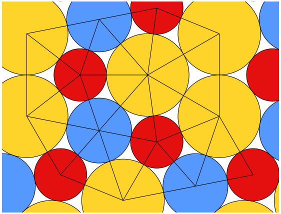


$$\forall\ \triangle,\ \ \delta_\triangle \leq \delta_{\triangle^*} = \delta^*$$

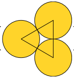
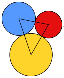
$$\delta = \sum_{\triangle \in \mathcal{T}} \delta_\triangle \leq \sum_{\triangle^* \in \mathcal{T}^*} \delta_{\triangle^*} \leq \delta^*$$

Idea of the proof for

Delaunay triangulation $\rightarrow$ weighted by the disc radii



Triangles have different densities:

$\delta($  $) \neq \delta($  $)$

What to do?

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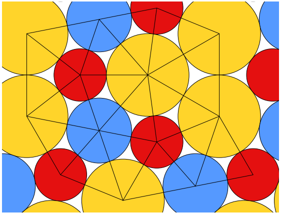

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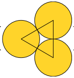
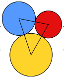
$\delta($  $) \neq \delta($  $)$

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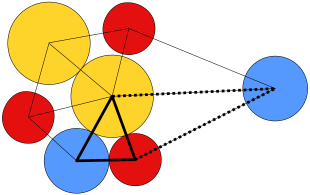

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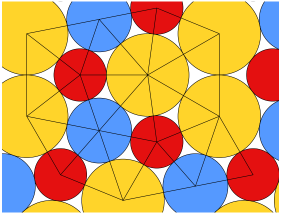

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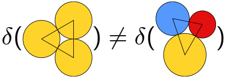

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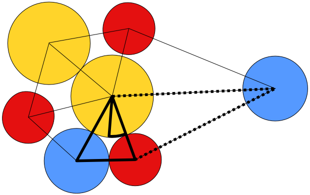
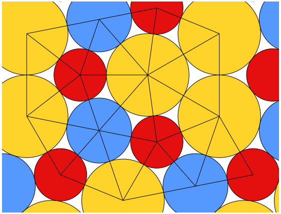


Some triangles “share their density” with neighbors

Delaunay triangulation → weighted by the disc radii



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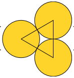
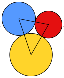
$\delta($  $) \neq \delta($  $)$

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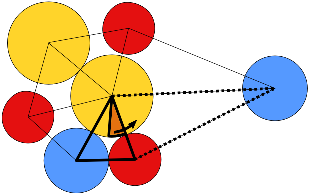

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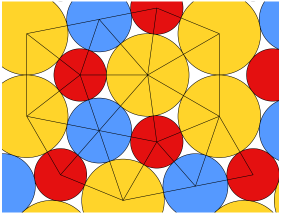

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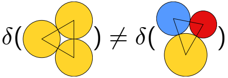

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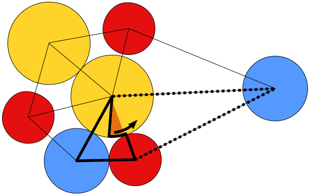

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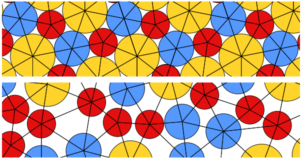
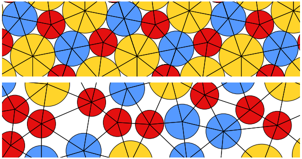
$\mathcal{T}^*$ – saturated triangulated packing of density δ

$\mathcal{T}$ – any other saturated packing with the same discs





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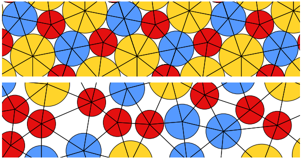
The **sparsity** of a triangle $\triangle \in \mathcal{T}$: $S(\triangle) = \delta \times area(\triangle) - cov(\triangle)$

$S(\triangle) > 0$ iff the density of covering of $\triangle$ is less than δ
$S(\triangle) < 0$ iff the density of covering of $\triangle$ is greater than δ

$$\delta(\mathcal{T}^*) \geq \delta(\mathcal{T}) \quad \Leftrightarrow \quad \sum_{\triangle \in \mathcal{T}} S(\triangle) \geq 0$$



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To show that, introduce a **potential** U such that for any triangle $\triangle \in \mathcal{T}$,

$$S(\triangle) \geq U(\triangle) \tag{$\triangle$}$$

and

$$\sum_{\triangle \in \mathcal{T}} U(\triangle) \geq 0 \tag{U}$$

(*U*): Instead of proving a **global** inequality

$$\sum_{\triangle \in \mathcal{T}} U(\triangle) \geq 0 \qquad (U)$$

we decompose $U(\triangle)$ into three vertex potentials: if A, B and C are the vertices of $\triangle$,

$$U(\triangle) = \dot{U}_{\triangle}^{A} + \dot{U}_{\triangle}^{B} + \dot{U}_{\triangle}^{C}$$

and prove a **local** inequality for each vertex $v \in \mathcal{T}$:

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$$4\dot{U}^v_{\triangle_1} + 2\dot{U}^v_{\triangle_2} + \dot{U}^v_{\triangle_3} = 0 \qquad \dot{U}^{v'}_{\triangle'_1} + \dot{U}^{v'}_{\triangle'_2} + \dot{U}^{v'}_{\triangle'_3} + \dot{U}^{v'}_{\triangle'_4} > 0$$

Delaunay triangulation properties $\to$ finite number of cases $\to$ verification by computer

Defining U, we try to make it as small as possible keeping it locally positive around any vertrex ($\bullet$).

How to check

$$S(\triangle) \geq U(\triangle) \qquad (\triangle)$$

on each triangle $\triangle$? (There is a continuum of them)

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Delaunay triangulation properties $\rightarrow$ uniform bound on edge length:

Verify $S(\triangle_{e_1,e_2,e_3}) \geq U(\triangle_{e_1,e_2,e_3})$ where

$$e_1 = [r_a+r_b, r_a+r_b+2s] \; e_2 = [r_c+r_b, r_c+r_b+2s] \; e_3 = [r_a+r_c, r_a+r_c+2s]$$

Not precise enough $\rightarrow$ dichotomy
(intervals intersect)

TODO

- classify all the remaining cases (modified Connelly's conjecture)

  for this: good comprehension of the density redistribution, more optimisation

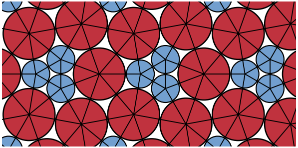
- maximal density for other disc sizes (which do not allow triangulated packings)

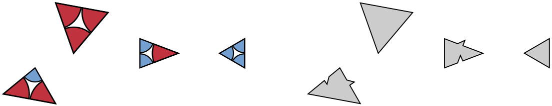
  deformations of triangulated packings keep the density high $\rightarrow$ good lower bound on the maximal density

- existence of a triangulated packing for a given set of discs – decidable?

  are there aperiodic triangulated packings?

triangulated packings $\sim$ tilings by triangles with local rules





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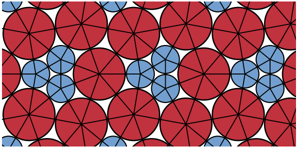
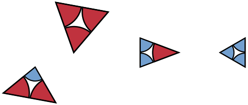
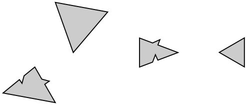
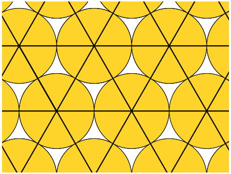




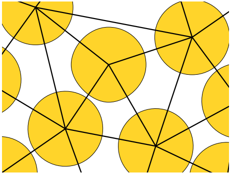


Thank you for your attention! :-)

A **Delaunay triangulation** of a packing: no points inside a circumscribed circle



$\delta^* = \delta_{\triangle^*} = \frac{\pi}{2\sqrt{3}}$



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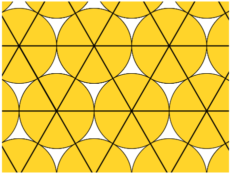

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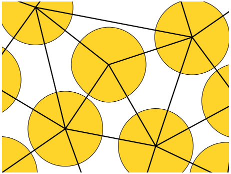

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The proof for

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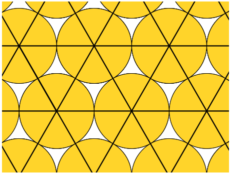

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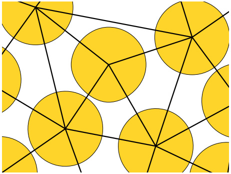

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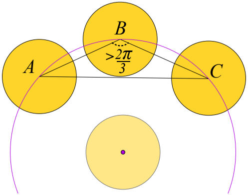

- The largest angle of any triangle is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ $\qquad R = \frac{|AC|}{2\sin\hat{B}} \geq \frac{1}{\sin\hat{B}}$

- The density of a triangle $\triangle$: $\delta_\triangle = \frac{\pi/2}{area(\triangle)}$

- The area of a triangle ABC with the largest angle $\hat{B}$ is $\frac{1}{2}|AB|\cdot|BC|\cdot\sin\hat{B}$ which is at least $\frac{1}{2}\cdot 2\cdot 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

- Thus the density of ABC is less or equal to $\frac{\pi/2}{\sqrt{3}}$

To store and perform computations on transcendental numbers (like π), we use intervals.

A representation of a number x is an interval I whose endpoints are exact values representable in a computer memory and such that $x \in I$.

```
sage: x = RIF(0,1)                                   # Interval [0,1]
sage: x < 2
True                                                 # ∀t ∈ [0,1], t < 2
sage: (x+x).endpoints()
(0.0, 2.0)                                           # [0,1]+[0,1]
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sage: Ipi = RIF(pi)                                   # Interval for π
(3.14159265358979, 3.14159265358980)
sage: sin(Ipi).endpoints()                            # Interval for sin(π)
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Intervals that intersect are incomparable.

```
sage: sin(Ipi) >= 0
False
sage: sin(Ipi) <= 0
False                                                 # Interval for sin(π) contains 0
sage: sin(Ipi) <= x
False                                                 # These intervals intersect
```