Algebraic Approach to Nivat's Conjecture

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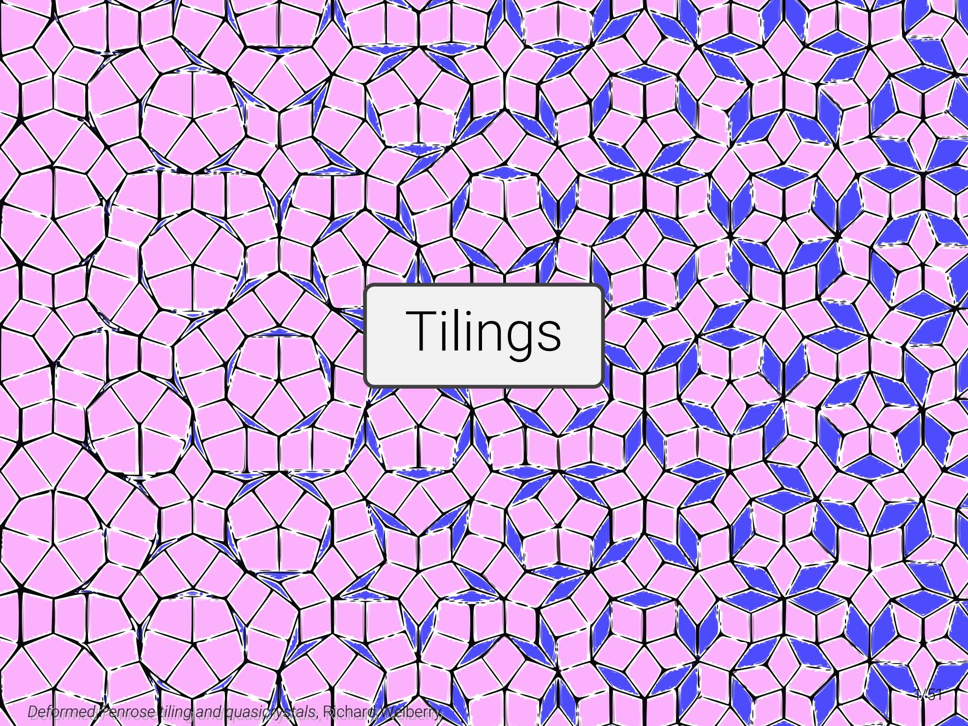
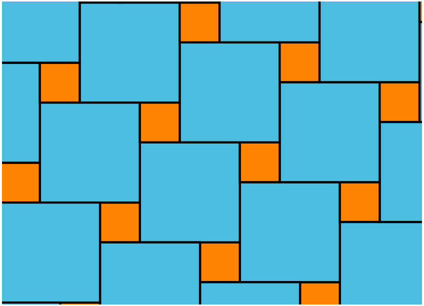
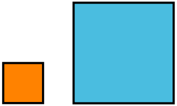


Tilings

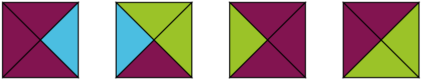
Deformed Penrose tiling and quasicrystals, Richard Welberry

Understanding **periodicity** just looking at the **tiles**





Wang tiles



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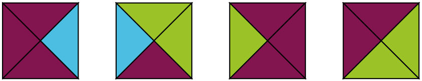
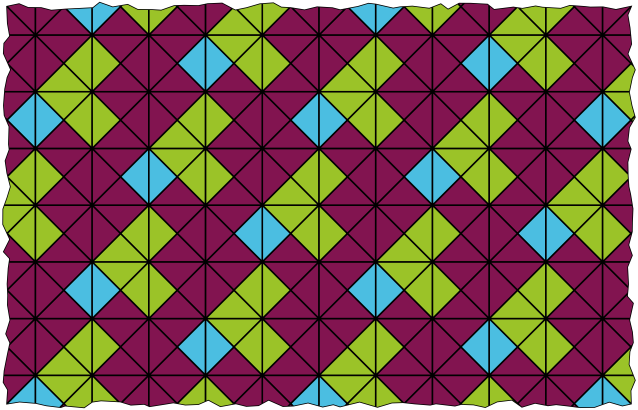


Wang tiles

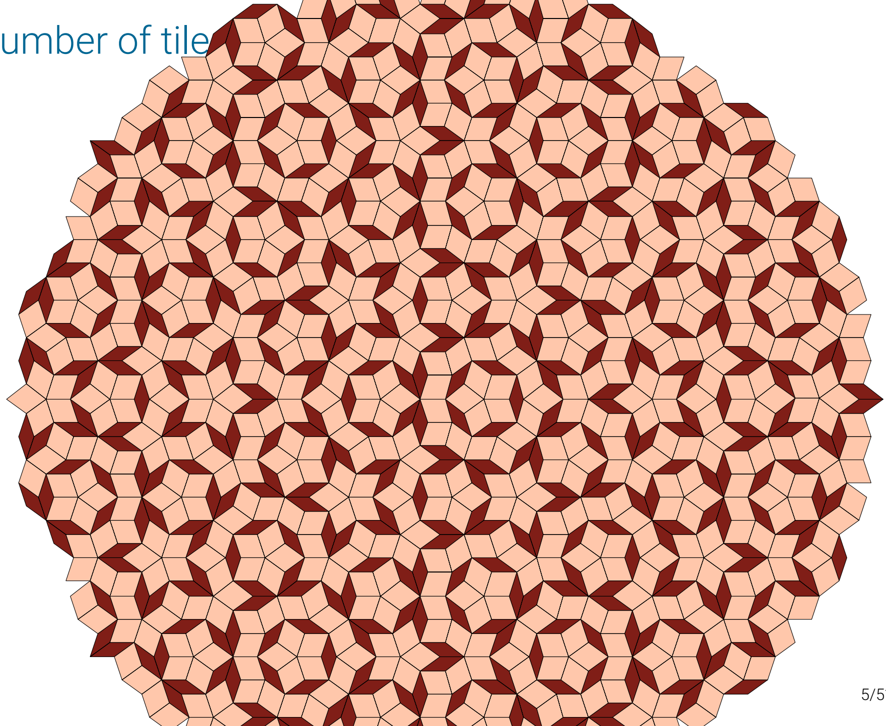


Number of tiles and (a)periodicity

Aperiodic sets of tiles:

- 20426 (Berger, 1966)
- 104 (Berger, 1964)
- 92 (Knuth, 1968)
- 56 (Robinson, 1967)
- 40 (Lauchli, 1975)
- 24 (Robinson, 1977)
- 16 (Ammann, 1978)
- 14 (Kari, 1996)
- 13 (Culik, 1996)
- **11 (Jeandel & Rao, 2015)** **Optimal!**

Number of tile



Number of tiles and (a)periodicity

Einstein problem: Is an aperiodic tiling *with one tile* possible?

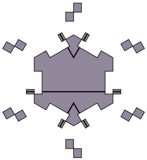
Number of tiles and (a)periodicity

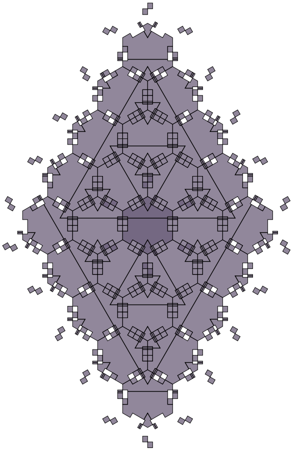
Ein Stein problem: Is an aperiodic tiling *with one tile* possible?

Number of tiles and (a)periodicity

Ein Stein problem: Is an aperiodic tiling *with one tile* possible?

Yes with a **Non connex** tile: Socolar & Taylor (2010)

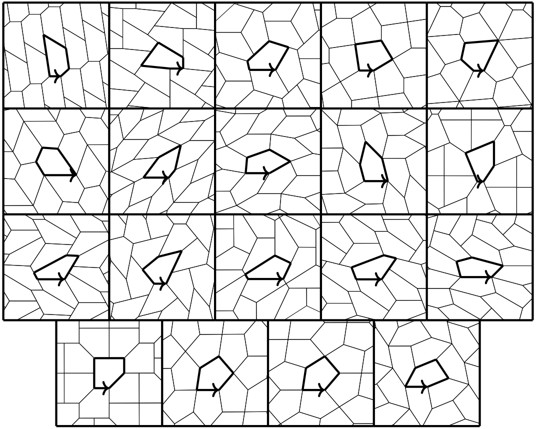




Number of tiles and (a)periodicity

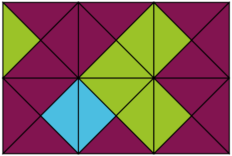
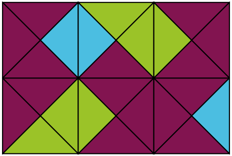
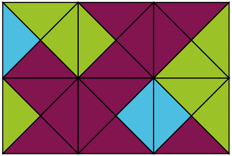
Ein Stein problem: Is an aperiodic tiling *with one tile* possible?

Connex tile: Cannot be a convex polygon (Rao, 2017)



Back to Wang tiles

We can count **small patterns** instead of single tiles

Back to Wang tiles

Pattern Complexity

$P_c(m,n) =$ number of rectangular patterns of size $m \times n$ in tiling c

Back to Wang tiles

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$P_c(m,n) =$ number of rectangular patterns of size $m \times n$ in tiling c

[Jeandel, Rao 2015] : $P_c(1,1) \leq 10 \Rightarrow c$ periodic (for Wang tilings)

Finally, the conjecture

What about any $P_c(m, n)$?

Conjecture (Nivat, 1997)

$$\exists m, n > 0, P_c(m, n) \leq mn \quad \Rightarrow \quad c \text{ periodic}$$

(Not only for Wang tilings)

Low complexity configurations

and Nivat's conjecture

Pattern complexity – 1D

$\mathcal{A} = \{$  $\}$

$w \in \mathcal{A}^{\mathbb{Z}}$

Pattern complexity – 1D

$$P_w(n) = \text{ number of patterns of size } n$$

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$$P_w(4) = 18$$

Pattern complexity – 1D

Theorem (Morse & Hedlund, 1938)

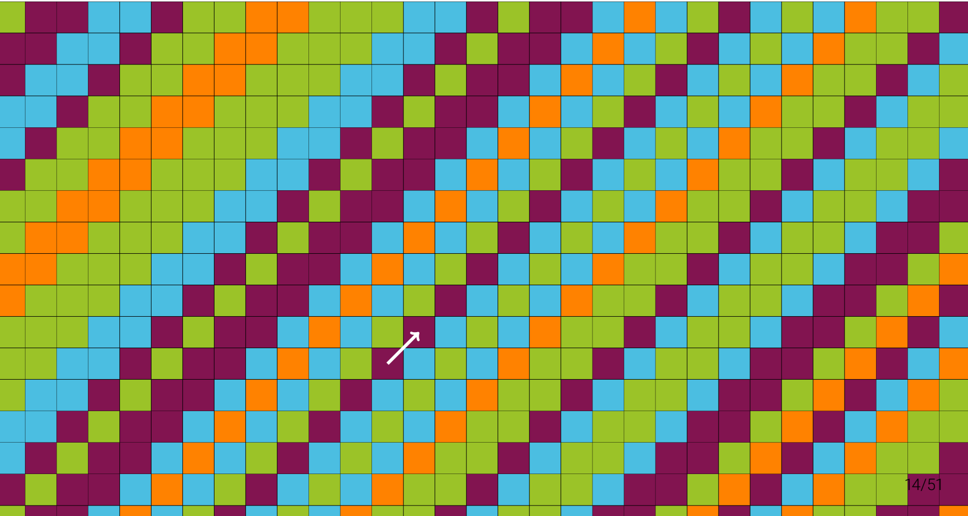
$$\forall w \in \mathcal{A}^{\mathbb{Z}},$$

$$\exists n > 0,\ P_w(n) \leq n \quad \Rightarrow \quad w \textit{ periodic}$$

Periodic configuration – 2D

$c \in \mathcal{A}^{\mathbb{Z}^2}$ is:

- **1-periodic** : $\exists \mathbf{u}, \forall \mathbf{v}, \ \ c_{\mathbf{v}-\mathbf{u}} = c_{\mathbf{v}}$



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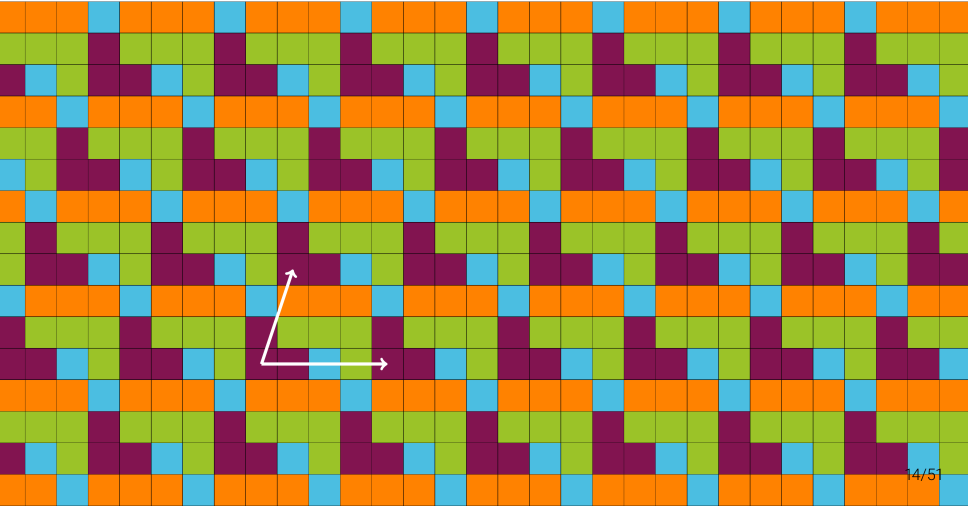
- **2-periodic** : c is 1-periodic along $\mathbf{u}_1, \mathbf{u}_2$ not colinear



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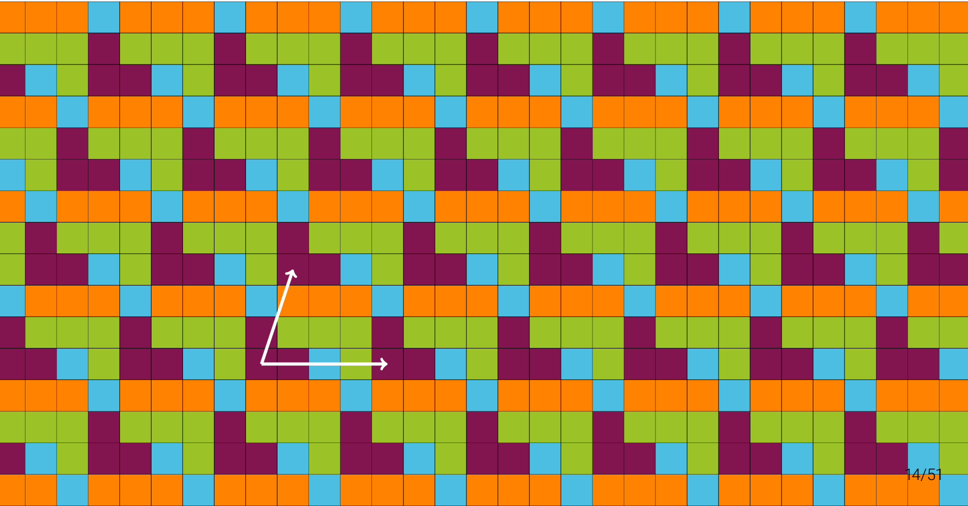
- **2-periodic** : c is 1-periodic along $\mathbf{u}_1, \mathbf{u}_2$ not colinear $\Rightarrow$ Finitely many different translations of c



Pattern complexity – 2D

$$P_c(m, n) = \text{ number of rectangular patterns of size } m \times n$$

2D: Nivat's conjecture

Conjecture (Nivat, 1997)

$$\forall c \in \mathcal{A}^{\mathbb{Z}^2},$$

$$\exists m, n > 0, P_c(m,n) \leq mn \quad \Rightarrow \quad c \text{ periodic}$$

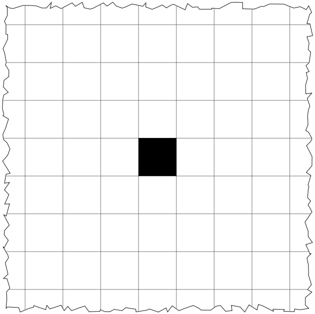
Better bound?

The bound is optimal:

Theorem

$$\exists c \in \mathcal{A}^{\mathbb{Z}^2},$$

$$\exists m, n > 0,\ P_c(m,n) = mn + 1 \quad \textit{and} \quad c \textit{ not periodic}$$



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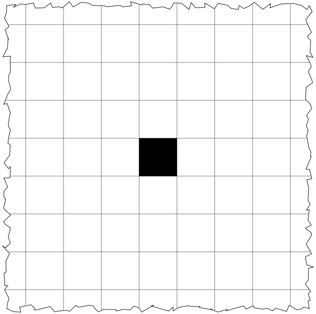

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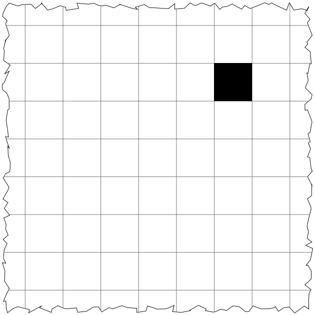

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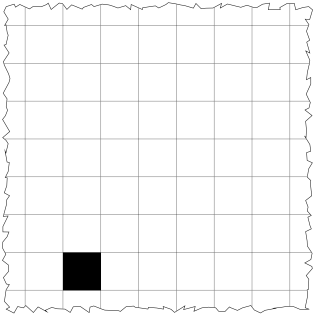

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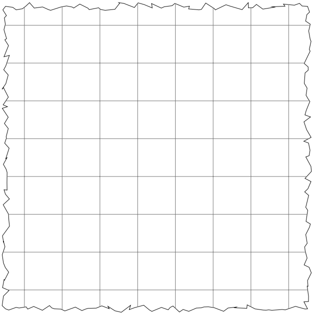


Higher dimension?

The same result does not hold in 3D and above:

$$\exists c \in \mathcal{A}^{\mathbb{Z}^3},$$
$$\exists n > 0,\ P_c(n,n,n) \leq n^3 \quad \textit{and} \quad c \textbf{ not } \textit{periodic}$$

Higher dimension?

1D – True but easy

2D – probably True but **very hard**!

3D – False

Higher dimension?

1D – True but easy

2D – probably True **very interesting**!

3D – False

To the conjecture

- $P_c(2,n) \leq 2n$ [Sanders & Tijdeman 2002]
- $P_c(3,n) \leq 3n$ [Cyr & Kra, 2016]

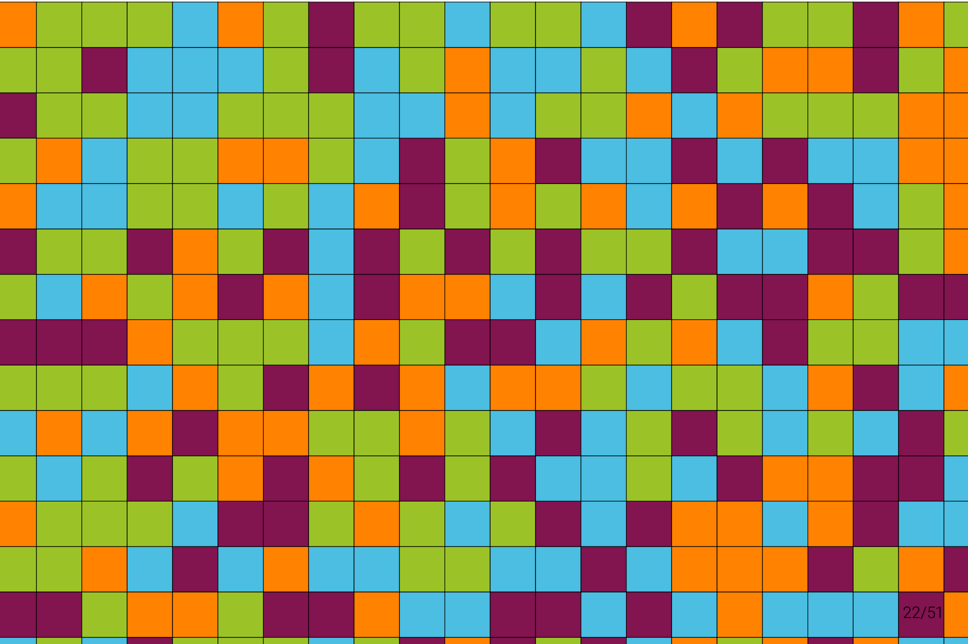
- $P_c(m,n) \leq \frac{mn}{144}$ [Epifanio, Koskas & Mignosi, 2003]
- $P_c(m,n) \leq \frac{mn}{16}$ [Quas & Zamboni, 2004]
- $P_c(m,n) \leq \frac{mn}{2}$ [Cyr & Kra, 2015]
- $P_c(m,n) \leq \frac{mn}{2} + |\mathcal{A}| - 1$ [Colle & Garibaldi, 2019]

- $P_c(m,n) \leq mn$ for infinitely many $m, n \Rightarrow c$ periodic [Kari & Szabados, 2015]

Algebraic approach

Kari and Szabados’ magical idea

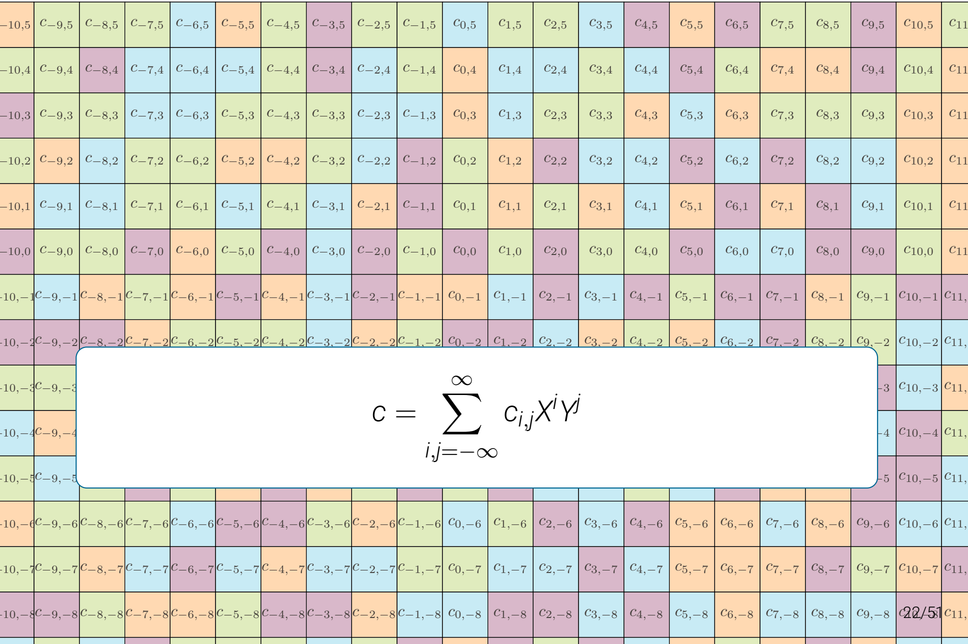
Configurations are (Laurent) series



Configurations are (Laurent) series

3	1	1	1	0	3	1	2	1	1	0	1	1	0	2	3	2	1	1	2	3	1
1	1	2	0	0	0	1	2	0	1	3	0	0	1	0	2	1	3	3	2	1	3
2	1	1	0	0	1	1	1	0	0	3	0	1	1	3	0	3	1	1	1	3	3
1	3	0	1	1	3	3	1	0	2	1	3	2	0	0	2	0	2	0	0	3	3
3	0	0	1	1	0	1	0	3	2	1	3	1	3	0	3	2	3	2	0	1	3
2	1	1	2	3	1	2	0	2	1	2	1	2	1	1	2	0	0	2	2	1	3
1	0	3	1	3	2	3	0	2	3	3	0	2	0	2	1	2	2	3	1	2	2
2	2	2	3	1	1	1	0	3	1	2	2	0	3	1	3	0	2	1	1	0	0
1	1	1	0	3	1	2	3	2	3	0	3	3	1	0	1	1	0	3	2	0	3
0	3	0	3	2	3	3	1	1	3	1	0	2	0	1	2	1	0	1	0	2	1
1	0	1	2	1	3	2	3	1	2	1	2	0	0	1	0	2	3	3	2	2	0
3	1	1	1	0	2	2	1	3	1	0	1	2	0	2	3	3	0	3	2	0	0
1	1	3	0	2	0	3	0	0	1	1	0	0	2	0	3	3	3	2	1	3	2
2	2	1	3	3	1	2	2	3	0	0	2	2	0	2	0	3	0	0	0	[illegible]	3

Configurations are (Laurent) series



$$c = \sum_{i,j=-\infty}^{\infty} c_{i,j} X^i Y^j$$

Operations: Sum

$$c + d = \sum_{i,j=-\infty}^{\infty} \left(c_{i,j} + d_{i,j}\right) X^i Y^j$$

Formal sum $\leftrightarrow$ Sum of configurations

Operations: Multiplication

$$X^a Y^b c = \sum_{i,j=-\infty}^{\infty} c_{i,j} X^{i+a} Y^{j+b}$$

Multiplication by $X^a Y^b \leftrightarrow$ Translation of vector (a, b)

Expressing periodicity

c (a, b)-periodic

$$\Leftrightarrow$$

$$X^a Y^b c = c$$

$$\Leftrightarrow$$

$$(X^a Y^b - 1)c = 0$$

Algebra is coming

$$\mathrm{Ann}(c) = \{p \mid pc = 0\}$$

$$c \text{ periodic } \Leftrightarrow \exists a, b \in \mathbb{Z}, (X^a Y^b - 1) \in \mathrm{Ann}(c)$$

$\mathrm{Ann}(c)$ is a **polynomial ideal**:

- $0 \in I$
- $f, g \in I \Rightarrow f + g \in I$
- $f \in I$ and h any polynomial $\Rightarrow fh \in I$

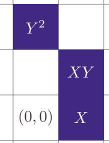
Warming up

c of low complexity – $\exists m, n, P_c(m,n) \leq mn$

Theorem (Kari & Szabados, 2015)

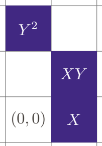
$\exists p \neq 0 \in \text{Ann}(c)$

Warming up (proof)



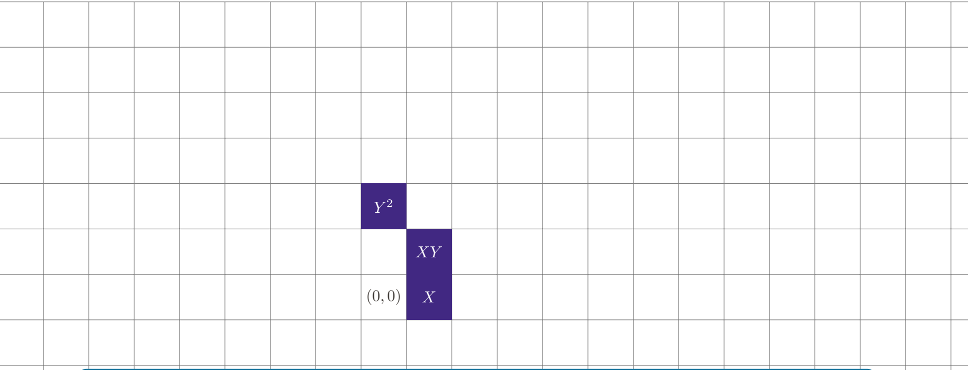
$$Xc + XYc + Y^2c = \mathbf{0}$$

Warming up (proof)



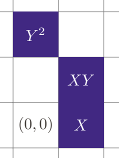
$$(Xc + XYc + Y^2c)_{(0,0)} = 0$$

Warming up (proof)



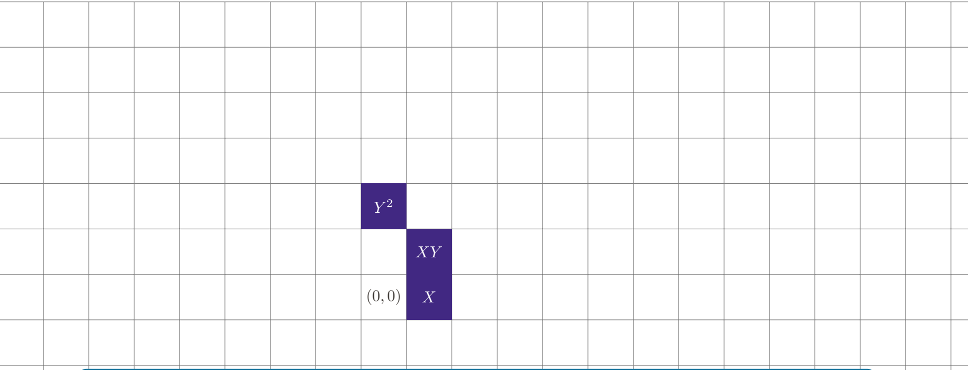
$$c_{1,0} + c_{1,1} + c_{0,2} = 0$$

Warming up (proof)



$$1c_{1,0} + 1c_{1,1} + 1c_{0,2} = 0$$

Warming up (proof)



$$p_{1,0}\ c_{1,0} + p_{1,1}\ c_{1,1} + p_{0,2}\ c_{0,2} = 0$$

Warming up (proof)

$p_{1,6}$	$p_{2,6}$	$p_{3,6}$	$p_{4,6}$	$p_{5,6}$	$p_{6,6}$	$p_{7,6}$	$p_{8,6}$
$p_{1,5}$	$p_{2,5}$	$p_{3,5}$	$p_{4,5}$	$p_{5,5}$	$p_{6,5}$	$p_{7,5}$	$p_{8,5}$
$p_{1,4}$	$p_{2,4}$	$p_{3,4}$	$p_{4,4}$	$p_{5,4}$	$p_{6,4}$	$p_{7,4}$	$p_{8,4}$
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$p_{1,1}$	$p_{2,1}$	$p_{3,1}$	$p_{4,1}$	$p_{5,1}$	$p_{6,1}$	$p_{7,1}$	$p_{8,1}$

$$p = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{i,j} X^i Y^j$$

Warming up (proof)

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Goal:

$$pc = \mathbf{0}$$

Warming up (proof)

$p_{1,6}$	$p_{2,6}$	$p_{3,6}$	$p_{4,6}$	$p_{5,6}$	$p_{6,6}$	$p_{7,6}$	$p_{8,6}$
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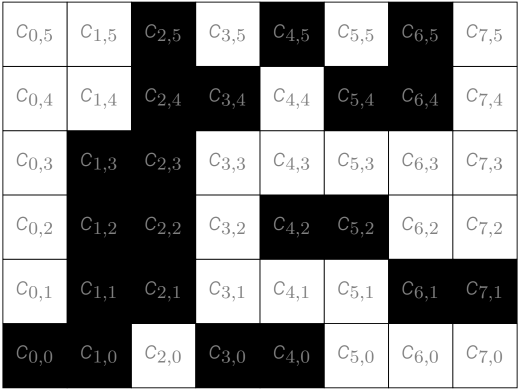
Goal:

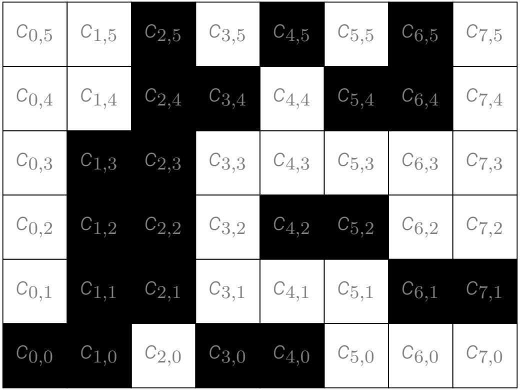
$$pc = \mathbf{B}$$

Warming up (proof)



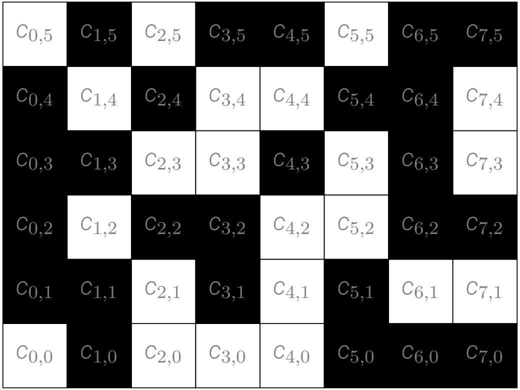
$$p_{0,0} \cdot c_{0,0} + p_{1,0} \cdot c_{1,0} + p_{2,0} \cdot c_{2,0} + p_{3,0} \cdot c_{3,0} + \ldots = b$$

Warming up (proof)



$$p_{0,0} \cdot 1 + p_{1,0} \cdot 1 + p_{2,0} \cdot 0 + p_{3,0} \cdot 1 + \ldots = b$$

Warming up (proof)



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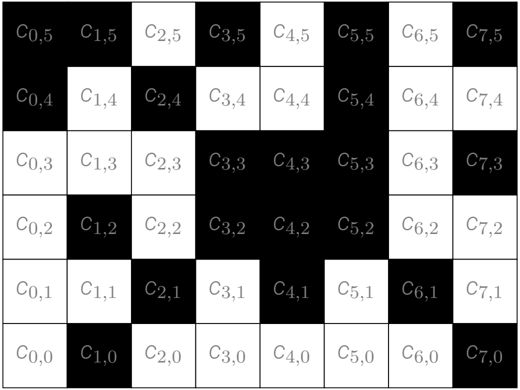
Warming up (proof)

$c_{0,5}$	$c_{1,5}$	$c_{2,5}$	$c_{3,5}$	$c_{4,5}$	$c_{5,5}$	$c_{6,5}$	$c_{7,5}$
$c_{0,4}$	$c_{1,4}$	$c_{2,4}$	$c_{3,4}$	$c_{4,4}$	$c_{5,4}$	$c_{6,4}$	$c_{7,4}$
$c_{0,3}$	$c_{1,3}$	$c_{2,3}$	$c_{3,3}$	$c_{4,3}$	$c_{5,3}$	$c_{6,3}$	$c_{7,3}$
$c_{0,2}$	$c_{1,2}$	$c_{2,2}$	$c_{3,2}$	$c_{4,2}$	$c_{5,2}$	$c_{6,2}$	$c_{7,2}$
$c_{0,1}$	$c_{1,1}$	$c_{2,1}$	$c_{3,1}$	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$	$c_{7,1}$
$c_{0,0}$	$c_{1,0}$	$c_{2,0}$	$c_{3,0}$	$c_{4,0}$	$c_{5,0}$	$c_{6,0}$	$c_{7,0}$

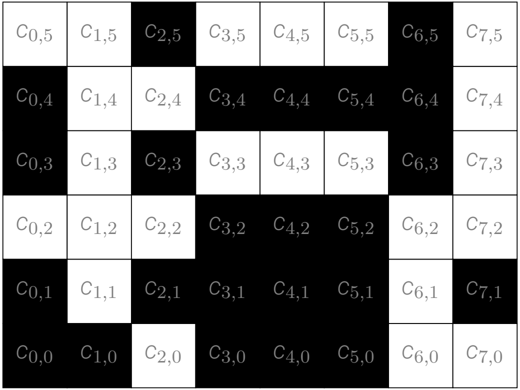
$$p_{0,0} \cdot 1 + p_{1,0} \cdot 0 + p_{2,0} \cdot 0 + p_{3,0} \cdot 1 + \ldots = b$$

Warming up (proof)



$$p_{1,0} + p_{7,0} + p_{2,1} + \ldots = b$$

Warming up (proof)



$$p_{0,0} + p_{1,0} + p_{0,1} + \ldots = b$$

Warming up (proof)

$\leq mn$ equations, $mn{+}1$ unknowns

$$\begin{cases} p_{0,0} + p_{1,0} + p_{3,0} + \ldots = b \\ p_{1,0} + p_{3,0} + p_{0,1} + \ldots = b \\ p_{0,0} + p_{3,0} + p_{4,0} + \ldots = b \\ p_{1,0} + p_{7,0} + p_{2,1} + \ldots = b \\ p_{0,0} + p_{1,0} + p_{0,1} + \ldots = b \\ \ldots \end{cases}$$

Warming up (proof)

$$\exists p, \quad pc = B$$

Warming up (proof)

$$\exists p, \quad pc = B$$

$$(X - 1)p \cdot c = 0$$

Warming up (more)

c of low complexity

Theorem (Kari & Szabados, 2015)

$\exists p \neq 0 \in \mathrm{Ann}(c)$

Warming up (more)

c of low complexity

Theorem (Kari & Szabados, 2015)

$\exists p \neq 0 \in \mathrm{Ann}(c)$

Theorem (Kari & Szabados, 2015)

$\exists a_1, b_1, a_2, b_2 \ldots, a_r, b_r \in \mathbb{Z}$

$$\left(X^{a_1} Y^{b_1} - 1\right)\left(X^{a_2} Y^{b_2} - 1\right) \cdots \left(X^{a_r} Y^{b_r} - 1\right) \in \mathrm{Ann}(c)$$

Warming up (more)

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Theorem (Kari & Szabados, 2015)

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$\rightarrow$ Hilbert's Nullstellensatz

More definitions

$\mathsf{V}(I) = \{x \in \mathbb{C}^d \mid \forall f \in I, f(x) = 0\}$	Common zeros of all $f \in I$
$\mathsf{I}(V) = \{f \in \mathbb{C}[X^{\pm 1}] \mid \forall x \in V, f(x) = 0\}$	Set of *f*s with all roots in V
$\mathsf{IV}(I)$	Set of *f*s with all roots in $\mathsf{V}(I)$

More definitions

$\mathrm{V}(I) = \{x \in \mathbb{C}^d \mid \forall f \in I, f(x) = 0\}$	Common zeros of all $f \in I$
$\mathrm{I}(V) = \{f \in \mathbb{C}[X^{\pm 1}] \mid \forall x \in V, f(x) = 0\}$	Set of fs with all roots in V
$\mathrm{IV}(I)$	Set of fs with all roots that are common zeros of all $f \in I$

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$$\sqrt{I} = \{f \mid \exists n, f^n \in I\}$$

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$\mathsf{IV}(I)$ Set of fs with all roots that are common zeros of all $f \in I$

$\sqrt{I} = \{f \mid \exists n, f^n \in I\}$

Line polynomial: $\exists a, b$ s.t. $p = \sum_i p_i X^{a \cdot i} Y^{b \cdot i}$

Proof (1)

1) Find a "nice" polynomial in IV(Ann(c))

$$g(X,Y) = \prod (X^{a_i} Y^{b_i} - 1) \in \mathrm{IV}(\mathrm{Ann}(c))$$

$$\Leftrightarrow$$

$g(X,Y) = 0$ for any common root (X,Y) of all $f \in$Ann(c)

$\rightarrow$ Not "too hard" to build

Proof (2)

2) Hilbert' Nullstellensatz: IV(I)$= \sqrt{I}$

So, $g \in \sqrt{\text{Ann}(c)}$

Proof (3)

3) Line polyomials are nice

$f_1^m, f_2^m, \ldots, f_k^m$ line polynomials

$$f_1^m \cdot f_2^m \cdots f_k^m \in \mathrm{Ann}(c) \;\Rightarrow\; f_1 \cdot f_2 \cdots f_k \in \mathrm{Ann}(c)$$

Proof (3)

3) Line polyomials are nice

$f_1^m, f_2^m, \ldots, f_k^m$ line polynomials

$$f_1^m \cdot f_2^m \cdots f_k^m \in \mathrm{Ann}(c) \;\Rightarrow\; f_1 \cdot f_2 \cdots f_k \in \mathrm{Ann}(c)$$

Exactly the case of g!

$$g \in \mathrm{Ann}(c)$$

$\square$

Warming up

c of low complexity – $\exists m, n, P_c(m, n) \leq mn$

Theorem (Kari & Szabados, 2015)

$\exists p \neq 0 \in \text{Ann}(c)$

Warming up

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Periodic Decomposition

c with a non-trivial annihilator (ex: c of low complexity)

Theorem (Kari & Szabados, 2015)

$\exists a_1, b_1, a_2, b_2 \ldots, a_r, b_r \in \mathbb{Z}$

$$\left(X^{a_1} Y^{b_1} - 1\right)\left(X^{a_2} Y^{b_2} - 1\right) \cdots \left(X^{a_r} Y^{b_r} - 1\right) \in \mathrm{Ann}(c)$$

Theorem (Kari & Szabados, 2015)

There exist periodic $c_1, \ldots, c_r$,

$$c = c_1 + \cdots + c_r$$

⚠ $c_1, \ldots, c_r$ may have infinite alphabet

Theorem (Kari & Szabados, 2015)

If c is such that $P_c(m,n) \leq mn$ for infinitely many m,n, then c is periodic

Theorem (Kari & Szabados, 2018)

If $c = c_1 + c_2$ with c_1 and c_2 periodic, then

$$P_c(m,n) \leq mn \Rightarrow c \text{ periodic} \qquad \text{(Nivat's conjecture)}$$

Uniform recurrent case

c **uniformly recurrent**: no patterns are "isolated"

Theorem (Kari & M., 2019)

If c is uniformly recurrent, then

$$P_c(m,n) \leq mn \Rightarrow c \text{ periodic} \qquad \text{(Nivat's conjecture)}$$

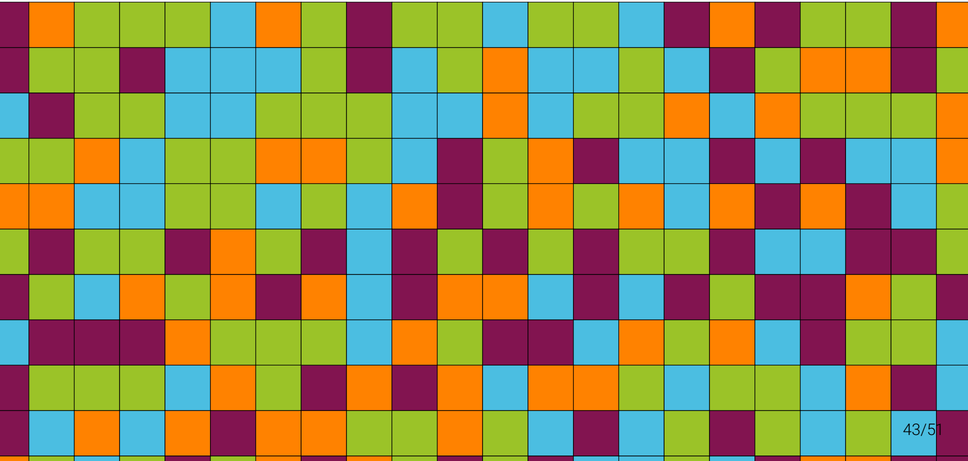
Subshifts

Linking complexity and (a)periodicity

Subshifts of Finite Type – Configurations

Finite alphabet: $\mathcal{A} = \{$  $\}$

Configuration: $c \in \mathcal{A}^{\mathbb{Z}^2}$



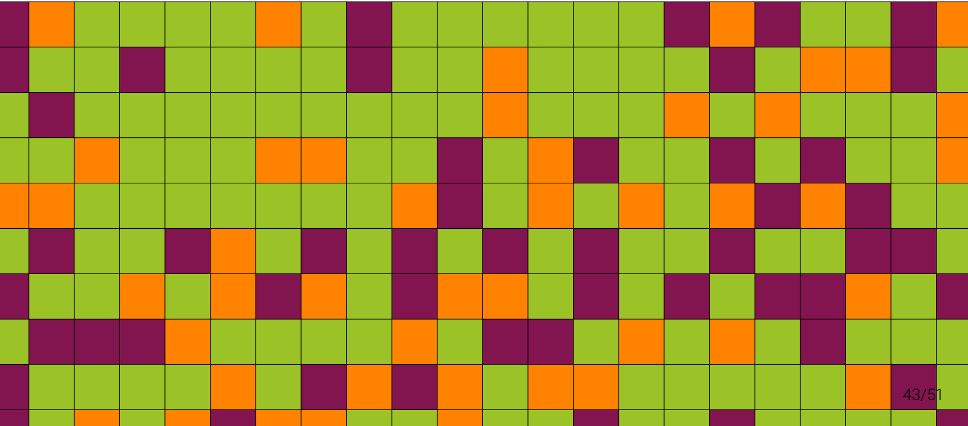
Subshifts of Finite Type

Finite alphabet: $\mathcal{A} = \{$  $\}$

Set of forbidden patterns: $F = \{$  $\}$

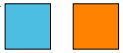
Subshift :

$$X_F = \{c \in \mathcal{A}^{\mathbb{Z}^2} \mid \forall m \in F, m \text{ does not appear in } c\}$$



Subshifts of Finite Type

Finite alphabet: $\mathcal{A} = \{$  $\}$

Set of forbidden patterns: $F = \{$  $\}$

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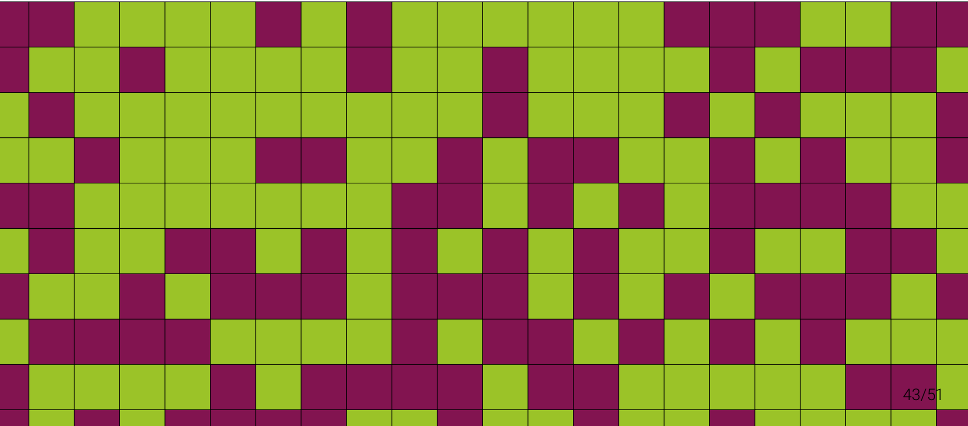

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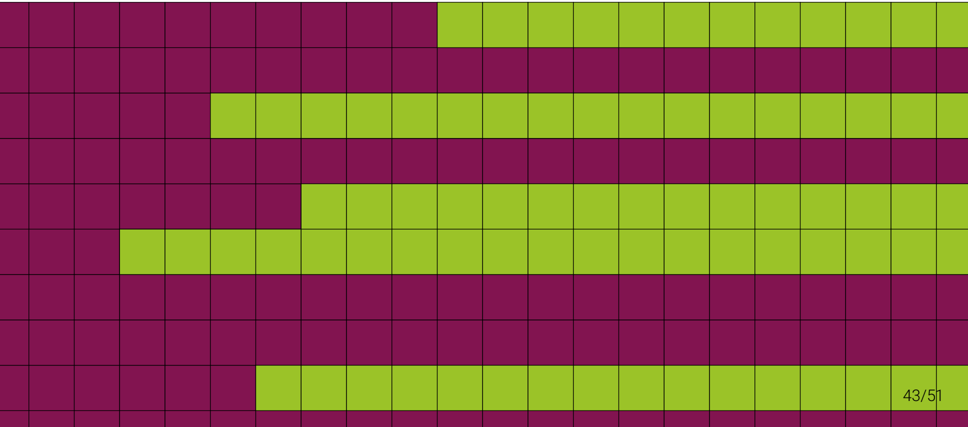

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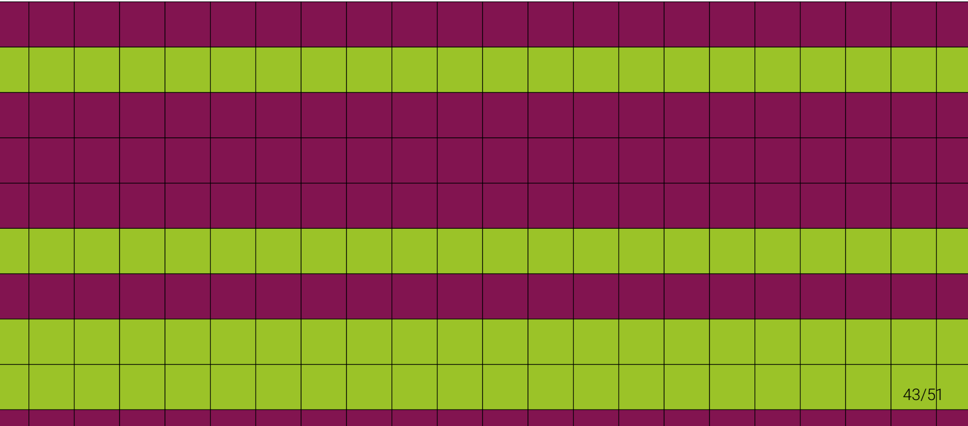

Subshifts of Finite Type

Finite alphabet: $\mathcal{A} = \{$  $\}$

Set of forbidden patterns: $F = \{$  $\}$

Subshift :

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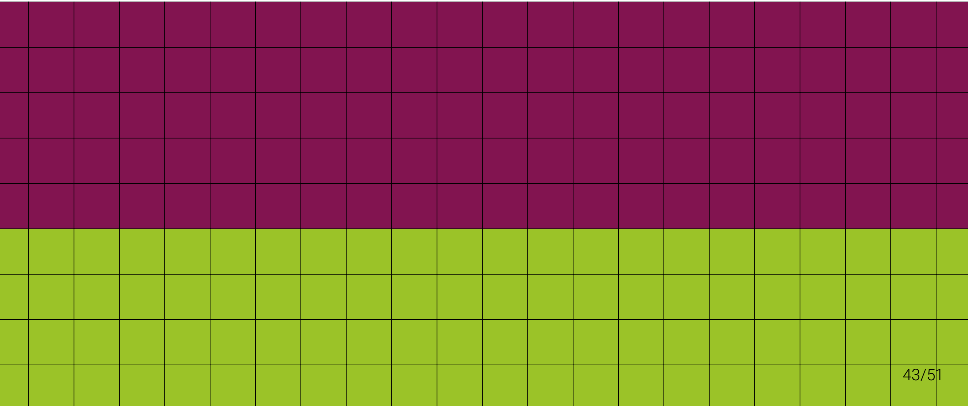


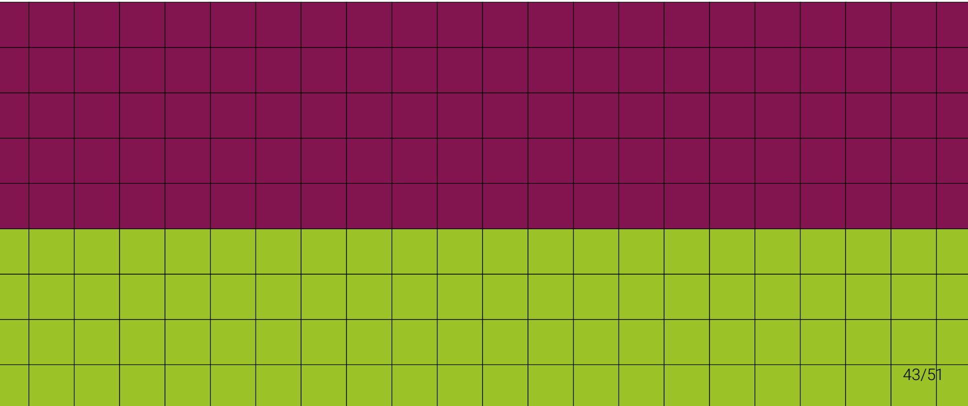
Subshifts of Finite Type

Finite alphabet: $\mathcal{A} = \{$  $\}$

Finite Set of forbidden patterns: $F = \{$  $\}$

Subshift of Finite Type (SFT):

$$X_F = \{c \in \mathcal{A}^{\mathbb{Z}^2} \mid \forall m \in F, m \text{ does not appear in } c\}$$



Aperiodicity of subshifts

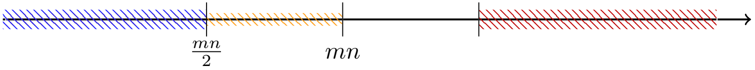
$\forall X, \forall c \in X, c$ periodic

$\forall X, \begin{array}{l} \exists c \in X, c \text{ periodic} \\ \exists c \in X, c \text{ not periodic} \end{array}$

$\exists X, \forall c \in X, c$ not periodic

Complexity and aperiodicity

$\forall X, \forall c \in X,\ c$ periodic — $\forall X, \exists c \in X,\ c$ periodic — $\exists X, \forall c \in X,\ c$ not periodic



Theorem (Cyr & Kra, 2015)

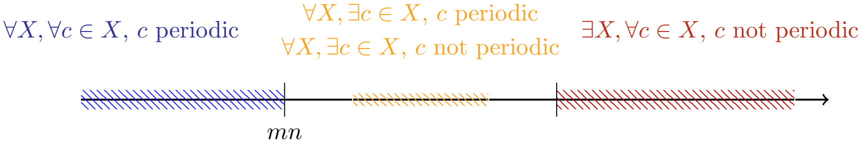
If $\exists m, n \in \mathbb{N}^*, P_c(m,n) \leq \frac{mn}{2}$, then c is periodic.

Theorem (Kari & M., 2019)

If $\exists m, n \in \mathbb{N}^*, c \in X$ s.t. $P_c(m,n) \leq mn$, then

$$\exists d \in X, d \text{ periodic}$$

Complexity and aperiodicity (and Nivat)



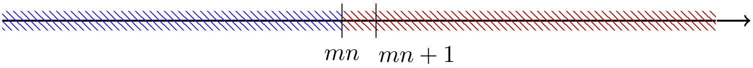
Conjecture (Nivat, 1997)

If $\exists m, n \in \mathbb{N}^*, P_c(m,n) \leq mn$, then c is periodic.

Cheating with the *size* of the local rule

$\forall X, \forall c \in X,\ c$ periodic

$\exists X, \forall c \in X,\ c$ not periodic



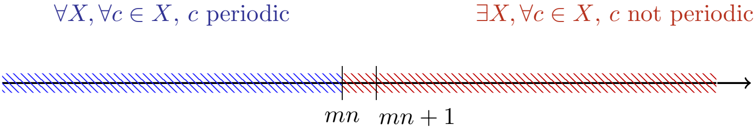
Theorem

For all m, n, there exists X an aperiodic SFT s.t.

$$\forall c \in X, P_c(m, n) = mn + 1$$

Proof:
Take any aperiodic tileset and *"blow it"* into $m \times n$ rectangles

Cheating with the *size* of the local rule

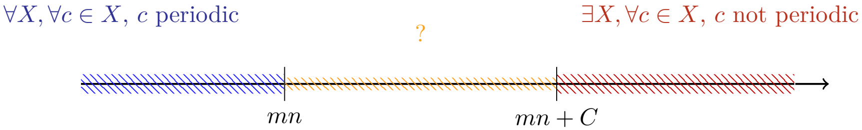


> **Theorem**
>
> For all m, n, there exists X an aperiodic SFT s.t.
>
> $$\forall c \in X, P_c(m,n) = mn + 1$$

Proof:
Take any aperiodic tileset and *"blow it"* into $m \times n$ rectangles

The complexity is computed with $m \times n$ **smaller** than the **size of the patterns**

With nearest-neighbor rules (Wang Tiles)

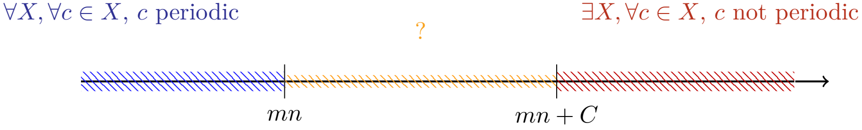


Theorem

There exists m, n, C, and X an aperiodic SFT s.t.

$$\forall c \in X, P_c(m,n) \leq mn + C$$

With nearest-neighbor rules (Wang Tiles)



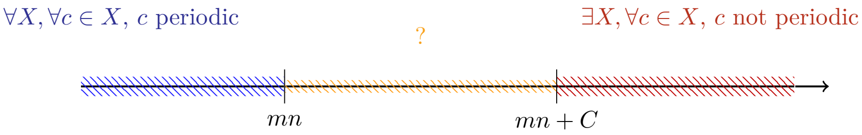
Theorem

There exists m, n, C, and X an aperiodic SFT s.t.

$$\forall c \in X, P_c(m,n) \leq mn + C$$

The complexity is computed with $m \times n$ **bigger** than the **size of the patterns**

With nearest-neighbor rules (Wang Tiles)



Theorem

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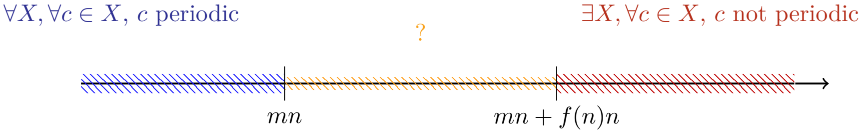
The complexity is computed with $m \times n$ **bigger** than the **size of the patterns**

Question

What is the smallest such C?

$C \geq 2$?

SFTs made of $m \times n$ patterns



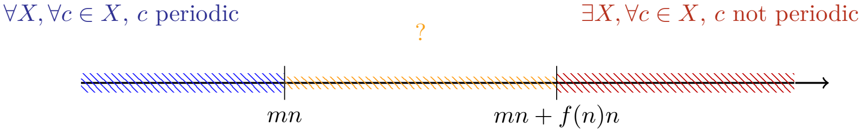
Theorem (Kari, 2020)

For any $f \notin O(1), \forall m, \exists n$, and X an aperiodic SFT s.t.

P is a set of $m \times n$ *allowed* patterns

$$|P| \leq mn + f(n)n$$

SFTs made of $m \times n$ patterns



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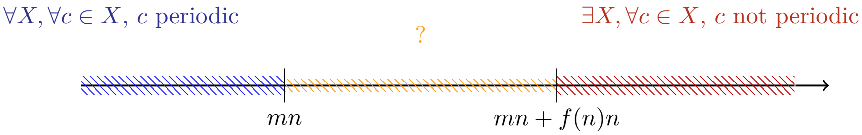
SFTs made of $m \times n$ patterns



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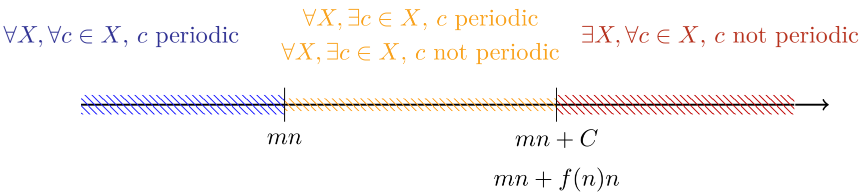
The complexity is computed with $m \times n$ the **size of the patterns**

Question

Better bound?



TODO 1

Does yellow region even exist?

TODO 2

Improve bounds

Thank you!

Anyone want to think about these questions with me? 😃😃