Surface entropies of subshifts of finite type

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**Definition 1**

2D Shifts

A 2D-*subshift* is a set of colorings $\mathbb{Z}^2 \rightarrow \Sigma$ that do not contain some family of forbidden patterns $\mathcal{F}$. Each family of forbidden patterns defines a subshift:

$$X_{\mathcal{F}} = \left\{ x \in \Sigma^{\mathbb{Z}^2} : \forall p \in \mathcal{F}, p \text{ does not appear in } x \right\}$$

**Definition 2**

Classification of shifts

1. A *subshift of finite type* (or SFT) is a subshift that can be defined by a finite family of forbidden patterns.

2. An *effective subshift* is a subshift that can be defined by a recursively enumerable family of forbidden patterns.
Example

\[ \mathcal{F} = \{ \text{cube}, \text{cylinder}, \text{pyramid} \} \]
\[ \mathcal{F} = \{ \text{example set} \} \]
$\mathcal{F} = \{ \text{image} \}$

Example
Example

$$\mathcal{F} = \left\{ \begin{array}{c}
\text{red}, \\
\text{blue}, \\
\text{blue}
\end{array} \right\}$$
The complexity function $N_n(X)$ of a shift $X$, for $n \in \mathbb{N}$, is defined as the number of different patterns of size $n \times n$ that appear in $X$.

$$\log N_n(X) \simeq hn^2 + h'n + \ldots$$

**Definition 3**

**Complexity function**

**Question:**

For an SFT, what are the possible values for $h'$?
Complexity function is density

\[ N_n = \# \text{ Different patterns of size } n \times n \]
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\[ N_n = \# \text{ Different patterns of size } n \times n \]

\[ \log N_n = \# \text{ white squares} \]
Topological entropy
What about topological entropies? (Part 1)

\[ \log N_n \approx h n^2 \]

[Hochman, Meyerovitch, 2010] proved that topological entropy were the \( \Pi_1 \) real numbers.

\[ \implies : h_{\text{top}} \in \Pi_1. \]

\[ \iff : \text{Let } h \in \Pi_1. \text{ We build an SFT with topological entropy } h. \]

**Arithmetical hierarchy of real numbers**

\( x \in \Pi_1 \) if there exists a recursively enumerable \( (r_k)_{k \in \mathbb{N}} \) in \( \mathbb{Q} \) such that:

\[ x = \inf_k r_k \]
What about topological entropies? (Part 2)

\[ h \approx 1.0100... \]
What about topological entropies? (Part 2)

\[ h = .10100... \]
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$\begin{align*}
h &= .10100... \\
\end{align*}$

Effective shift!
What about topological entropies? (Part 2)

\[ h = .10100... \]
What about topological entropies? (Part 2)

$h = .10100...$
What about topological entropies? (Part 2)

By [Hochman, 2010], [Aubrun, Sablik, 2013], [Durand, Romashchenko, Shen, 2012]:

Effective 1D $\simeq$ SFT 2D

$$h = .10100...$$
What about topological entropies? (Part 2)

By Hochman, 2010, Aubrun, Sablik, 2013, Durand, Romashchenko, Shen, 2012:

Effective 1D $\approx$ SFT 2D

How many white squares?

$h = .10100...$
What about topological entropies? (Part 2)

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- Effective 1D ≃ SFT 2D
What about topological entropies? (Part 2)

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$h = .10100...$
What about topological entropies? (Part 3)

\[ \log N_n \sim h n^2 \quad \Rightarrow \quad h_{\text{top}} = h \]
Surface entropy
Surface entropies (Part 1)

\[ \log N_n \simeq hn^2 + h'n \]

\[ \implies \text{: Surface entropies are } \Pi_3 \text{ real numbers.} \]

\[ \iff \text{: Let } h' \in \Pi_3, \]

\[ h' = \limsup_{k \to +\infty} r_k \]

We create an SFT with surface entropy \( h' \).

**Arithmetical hierarchy of real numbers**

\( x \in \Pi_3 \) if there exists a recursively enumerable \((r_k)_{k \in \mathbb{N}}\) in \( \Pi_1 \) such that:

\[ x = \limsup_k r_k \]
Surface entropies: the sparse squares (Part 2)
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\[ r^4 = \frac{1}{2} \]
Surface entropies: the sparse squares (Part 2)

\[ r_4 = \frac{1}{2} \]
This is an SFT!
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Surface entropies: structure (Part 3)

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Surface entropies: structure (Part 3)

$r_1$

$r_2$

$r_3$

$h_s = \lim \sup r_k = h'$

This is an SFT!
Surface entropies: structure (Part 3)

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This is an SFT!

\[ h_s = \lim \sup r_k = h' \]
Surface entropies: structure (Part 3)

\[ h_s = \limsup_{k} r_k = h' \]
Surface entropies: structure (Part 3)

This is an SFT!
Surface entropies: structure (Part 3)

$h_s = \lim \sup_{n \to \infty} \frac{1}{n} \log r_n$

This is an SFT!
Conclusion
**Question:**

\[ \log N_n \simeq h n^2 + h'n \]

In an SFT, what are the possible values for \( h' \)?

**Answer:**

Surface entropies are exactly the class of \( \Pi_3 \) real numbers!
Questions?