Structure of multidimensional constant-shape substitutions

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Journées SDA2 2020 : Systèmes Dynamiques, Automates & Algorithmes

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1. Definitions and basic properties

2. Homomorphisms between multidimensional constant-shape substitutions
   - Consequences of Main Theorem and Examples

Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be an expansion linear map, i.e, $\|L\| > 1$ and $\|L^{-1}\| < 1$ such that $L(\mathbb{Z}^d) \subseteq \mathbb{Z}^d$. Let $F = L([0,1)^d) \cap \mathbb{Z}^d$, and $\mathcal{A}$ be a finite alphabet.

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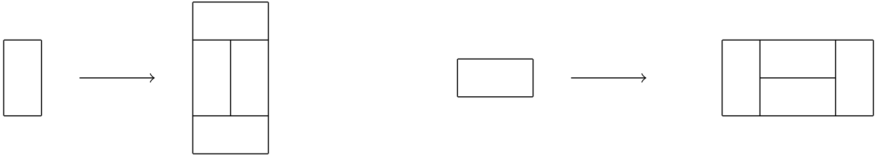
Every $\mathbf{p} \in \mathbb{Z}^d$ can be written in a unique way as $\mathbf{p} = L(\mathbf{j}) + \mathbf{k}$, with $\mathbf{j} \in \mathbb{Z}^d$ and $\mathbf{k} \in F$. We consider the substitution ζ as a map from $\mathcal{A}^{\mathbb{Z}^d}$ to itself given by

$$\zeta(x)_{L(\mathbf{j})+\mathbf{k}} = \zeta(x(\mathbf{j}))_{\mathbf{k}}.$$

Example of a multidimensional constant-shape substitution:
$L = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $F = [\![0,1]\!] \times [\![0,2]\!]$ and

$$a \mapsto \begin{matrix} b & c \\ c & b \\ a & b \end{matrix} \qquad b \mapsto \begin{matrix} a & c \\ c & b \\ b & c \end{matrix} \qquad c \mapsto \begin{matrix} c & b \\ a & c \\ c & b \end{matrix}.$$

The table tiling:



For any $n > 0$ the n-th iteration is defined as $\zeta^n : \mathcal{A} \to \mathcal{A}^{F_n^\zeta}$, with $F_{n+1}^\zeta = L_\zeta(F_n^\zeta) + F_1^\zeta$.

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$$\mathcal{L}_\zeta = \{\mathtt{p} \sqsubseteq \zeta^n(a),\ \text{for some } n > 0,\ a \in \mathcal{A}\}.$$

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With the language we define the subshift $(X_\zeta, S, \mathbb{Z}^d)$ as the set of all sequences $x \in \mathcal{A}^{\mathbb{Z}^d}$ such that every pattern of x is in $\mathcal{L}_\zeta$.

The system $(X_\zeta, T, \mathbb{Z}^d)$ is uniquely ergodic, i.e., there exists an unique T-invariant measure μ_ζ (which is also ergodic).

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P. Michel (1974) proved the one-dimensional case.
B. Solomyak (1997) proved it for self-affine tilings with the $\mathbb{R}^d$-action
J.-Y. Lee, R. Moody, B. Solomyak (2003) proved it for Substitution Delone sets with $\mathbb{R}^d$-actions.

A **homomorphism** is a continuous surjective map $\phi : X \to Y$ that commutes with the action, i.e., $T^{\mathbf{n}} \circ \phi(x) = \phi \circ T^{\mathbf{n}}(x)$ for all $x \in X$ and $\mathbf{n} \in \mathbb{Z}^d$. We denote the set of all homomorphisms as $\mathrm{Hom}(X, Y, T, \mathbb{Z}^d)$.

If $X = Y$, ϕ is called an **endomorphism** and if ϕ is invertible, ϕ is called an **automorphism**. We denote the set of all automorphisms as $\mathrm{Aut}(X, T, \mathbb{Z}^d)$.

We use $m\,\mathrm{Hom}$ or $m\,\mathrm{Aut}$ for measurable maps.

It is well known that homomorphisms between symbolic systems are characterized:

Theorem (Curtis-Hedlund-Lyndon)

Let $(X, S, \mathbb{Z}^d)$ and $(Y, S, \mathbb{Z}^d)$ be two subshifts. A map $\phi : (X, S, \mathbb{Z}^d) \to (Y, S, \mathbb{Z}^d)$ is a factor map if and only if there exists a $B(0, r)$-block map $\Phi : \mathcal{L}_{B(0,r)}(X) \to \mathcal{L}_1(Y)$, such that $\phi(x)_{\mathbf{n}} = \Phi(x|_{\mathbf{n}+B(0,r)})$, for all $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$ (i.e., ϕ is a sliding block code).

Theorem (Main Theorem)

Let $(X_{\zeta_1}, S, \mathbb{Z}^d)$, $(X_{\zeta_2}, S, \mathbb{Z}^d)$ be two substitution dynamical system with ζ_1, ζ_2 having the same expansion map L. If $(X_{\zeta_2}, S, \mathbb{Z}^d)$ does not have purely discrete spectrum, then for every $\phi \in m\,\mathrm{Hom}(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$ there exists $\mathbf{j} \in \mathbb{Z}^d$ with $S^{\mathbf{j}}\phi$ equals to $\psi \in \mathrm{Hom}(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$ μ_{ζ_1}-a.e., satisfying the following two properties:

1. *ψ is a sliding block code of radius $\sqrt{d}$.*
2. *There exist an integer $n > 0$ and $\mathbf{p} \in \mathbb{Z}^d$ such that $S^{\mathbf{p}}\psi\zeta_1^n = \zeta_2^n\psi$.*

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2. *There exist an integer $n > 0$ and $\mathbf{p} \in \mathbb{Z}^d$ such that $S^{\mathbf{p}}\psi\zeta_1^n = \zeta_2^n\psi$.*

This was first proved by B. Host and F. Parreau (1989) for constant-length in the one-dimensional case.

Corollary

Let ζ be an aperiodic multidimensional constant-shape substitution with spectrum not purely discrete. Then $(X_\zeta, S, \mathbb{Z}^d)$ is coalescent.

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Let $(X_\zeta, S, \mathbb{Z}^d)$ be a subshift from aperiodic substitution ζ with not purely discrete spectrum. Then $\mathrm{Aut}(X_\zeta, S, \mathbb{Z}^d)$ is virtually $\mathbb{Z}^d$, i.e., $|\,\mathrm{Aut}(X_\zeta, \mathbb{Z}^d)/\left\langle S \right\rangle| < \infty$.

Both results are also consequences of the following result

Proposition

Let ζ be a substitution. Then, there exist finitely many ζ-invariant orbits. The bound is an explicit formula which only depends on $|\mathcal{A}|$, $\|L_{\zeta}^{-1}\|$ and $\det(L_{\zeta} - \mathrm{id})$.

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An orbit $\mathcal{O}(x, \mathbb{Z}^d)$ is called ζ-**invariant** if there exists $\mathbf{j} \in \mathbb{Z}^d$ such that $\zeta(x) = S^{\mathbf{j}}x$ (The definition is independent of the choice of the point in the orbit of x).

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Remark

If $L_\zeta = 2\,\mathrm{id}_{\mathbb{R}^d}$ the only ζ-invariant orbits are the ones given by the fixed points of the substitution.

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This was first proved by F. Durand (2000) for the one-dimensional case.

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The **repetitivity function** of a substitution is the map $M_{X_\zeta} : \mathbb{R}_+ \to \mathbb{R}_+$ defined for $R > 0$ as the smallest radius such that every ball $B(\mathbf{n}, M_{X_\zeta}(R))$ contains an occurrence of every pattern with $\text{diam}(\text{supp}(\mathtt{p})) \leq 2R$.

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We say that the substitution is **linearly recurrent** or **linearly repetitive** if the repetitivity function has a linear growth, i.e., there exists $C > 0$ such that $M_{X_\zeta}(R) \leq C \cdot R$.

Example of a non linearly recurrent substitution: $L = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $F = [\![0,1]\!] \times [\![0,2]\!]$ and

$$a \mapsto \begin{matrix} b & c \\ c & b \\ a & b \end{matrix} \qquad b \mapsto \begin{matrix} a & c \\ c & b \\ b & c \end{matrix} \qquad c \mapsto \begin{matrix} c & b \\ a & c \\ c & b \end{matrix}.$$

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Lemma

Let ζ be an aperiodic primitive substitution, then $M_{X_\zeta}(R)$ is $\mathcal{O}\left(R^{-\frac{\log(\|L_\zeta\|)}{\log\left(\|L_\zeta^{-1}\|\right)}}\right)$.

Remark

1. *If the expansion map is symmetric, then* $M_{X_\zeta}(R) = \mathcal{O}\left(R^{\frac{\log(|\lambda_1|)}{\log(|\lambda_d|)}}\right)$, *where* $|\lambda_1|$, $|\lambda_d|$ *are the maximum and minimum of the absolute values of the eigenvalues of* L_ζ, *respectively.*

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1. *If the expansion map is symmetric, then* $M_{X_\zeta}(R) = \mathcal{O}\left(R^{\frac{\log(|\lambda_1|)}{\log(|\lambda_d|)}}\right)$, *where* $|\lambda_1|$, $|\lambda_d|$ *are the maximum and minimum of the absolute values of the eigenvalues of* L_ζ, *respectively.*
2. *In the case of a self-similar tiling, i.e.* $\|L_\zeta\| = \|L_\zeta^{-1}\|^{-1} = \lambda$, *substitutions are linearly recurrent.*

THE END