Algebraic relations between values of Mahler functions in positive characteristic

Gwladys Fernandes

Université de Versailles Saint-Quentin en Yvelines

December 3, 2020

Definition 1.1

1. *A **number field** $\mathbb{K}$ is a finite extension of $\mathbb{Q}$. That is a field which contains $\mathbb{Q}$ and of finite dimension, as a $\mathbb{Q}$-vector space.*

Definition 1.1

1. *A **number field** $\mathbb{K}$ is a finite extension of $\mathbb{Q}$. That is a field which contains $\mathbb{Q}$ and of finite dimension, as a $\mathbb{Q}$-vector space.*
2. *We say that $x \in \mathbb{C}$ is **algebraic** over $\mathbb{K}$ if there exists a non-zero polynomial $P(X) \in \mathbb{K}[X]$ such that $P(x) = 0$. Otherwise, we say that x is **transcendental** over $\mathbb{K}$.*

Definition 1.1

1. *A **number field** $\mathbb{K}$ is a finite extension of $\mathbb{Q}$. That is a field which contains $\mathbb{Q}$ and of finite dimension, as a $\mathbb{Q}$-vector space.*
2. *We say that $x \in \mathbb{C}$ is **algebraic** over $\mathbb{K}$ if there exists a non-zero polynomial $P(X) \in \mathbb{K}[X]$ such that $P(x) = 0$. Otherwise, we say that x is **transcendental** over $\mathbb{K}$.*
3. *We let $\overline{\mathbb{K}}$ be the set all the **algebraic** complex numbers over $\mathbb{K}$.*

Definition 1.1

1. *A **number field** $\mathbb{K}$ is a finite extension of $\mathbb{Q}$. That is a field which contains $\mathbb{Q}$ and of finite dimension, as a $\mathbb{Q}$-vector space.*
2. *We say that $x \in \mathbb{C}$ is **algebraic** over $\mathbb{K}$ if there exists a non-zero polynomial $P(X) \in \mathbb{K}[X]$ such that $P(x) = 0$. Otherwise, we say that x is **transcendental** over $\mathbb{K}$.*
3. *We let $\overline{\mathbb{K}}$ be the set all the **algebraic** complex numbers over $\mathbb{K}$.*
4. *We say that $x_1, \ldots, x_n \in \mathbb{C}$ are **algebraically dependent** over $\mathbb{K}$ if there exists a non-zero polynomial $P(X_1, \ldots, X_n) \in \mathbb{K}[X_1, \ldots, X_n]$ such that $P(x_1, \ldots, x_n) = 0$.*

Definition 1.2

Let $d \geq 2$ be an integer. Let $\mathbb{K}$ be a number field. Let $f(z) \in \mathbb{K}\{z\}$ be a convergent power series in a neighbourhood of the origin, with coefficients in $\mathbb{K}$.

Definition 1.2

Let $d \geq 2$ be an integer. Let $\mathbb{K}$ be a number field. Let $f(z) \in \mathbb{K}\{z\}$ be a convergent power series in a neighbourhood of the origin, with coefficients in $\mathbb{K}$.

*Then, $f(z)$ is d-**Mahler** over $\mathbb{K}(z)$ if there exist polynomials $P_0(z), \ldots, P_n(z) \in \mathbb{K}[z]$, $P_n(z) \neq 0$, such that:*

$$P_0(z)f(z) + P_1(z)f(z^d) + \cdots + P_n(z)f(z^{d^n}) = 0.$$

Definition 1.2

Let $d \geq 2$ be an integer. Let $\mathbb{K}$ be a number field. Let $f(z) \in \mathbb{K}\{z\}$ be a convergent power series in a neighbourhood of the origin, with coefficients in $\mathbb{K}$.

*Then, $f(z)$ is d-**Mahler** over $\mathbb{K}(z)$ if there exist polynomials $P_0(z), \ldots, P_n(z) \in \mathbb{K}[z]$, $P_n(z) \neq 0$, such that:*

$$P_0(z)f(z) + P_1(z)f(z^d) + \cdots + P_n(z)f(z^{d^n}) = 0.$$

Example: $f(z) = \sum_{n=0}^{+\infty} z^{2^n}$.

Definition 1.2

Let $d \geq 2$ be an integer. Let $\mathbb{K}$ be a number field. Let $f(z) \in \mathbb{K}\{z\}$ be a convergent power series in a neighbourhood of the origin, with coefficients in $\mathbb{K}$.

*Then, $f(z)$ is d-**Mahler** over $\mathbb{K}(z)$ if there exist polynomials $P_0(z), \ldots, P_n(z) \in \mathbb{K}[z]$, $P_n(z) \neq 0$, such that:*

$$P_0(z)f(z) + P_1(z)f(z^d) + \cdots + P_n(z)f(z^{d^n}) = 0.$$

Example: $f(z) = \sum_{n=0}^{+\infty} z^{2^n}$.

$$f\left(z^2\right) = f(z) - z.$$

Definition 1.3

*We say that the column vector whose coordinates are the power series $f_1(z), \ldots, f_n(z) \in \mathbb{K}[[z]]$ satisfies a d-**Mahler system** if there exists a matrix $A(z) \in GL_n(\mathbb{K}(z))$ such that*

$$\begin{pmatrix} f_1(z^d) \\ \vdots \\ f_n(z^d) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix}. \tag{1}$$

Problem

Problem

Let $f(z) \in \mathbb{K}\{z\}$, and $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$ be d-Mahler functions, and let $\alpha \in \overline{\mathbb{K}}$.

Problem

Let $f(z) \in \mathbb{K}\{z\}$, and $f_1(z), \dots, f_n(z) \in \mathbb{K}\{z\}$ be d-Mahler functions, and let $\alpha \in \overline{\mathbb{K}}$.

We are interested in results of transcendence of $f(\alpha)$ or algebraic independence of $f_1(\alpha), \dots, f_n(\alpha)$ over $\overline{\mathbb{K}}(z)$.

Definition 1

Let $k \geq 2$ be an integer. A sequence $(a_n)_n$ is k-**automatic** if it is generated by a finite k-automaton: a machine which reads, for every integer n, the sequence of the expansion of n in base k and returns the value a_n.

Definition 1

Let $k \geq 2$ be an integer. A sequence $(a_n)_n$ is k-**automatic** if it is generated by a finite k-automaton: a machine which reads, for every integer n, the sequence of the expansion of n in base k and returns the value a_n.

Example : the Thue-Morse sequence is a 2-automatic sequence.

Definition 1

Let $k \geq 2$ be an integer. A sequence $(a_n)_n$ is k-**automatic** if it is generated by a finite k-automaton: a machine which reads, for every integer n, the sequence of the expansion of n in base k and returns the value a_n.

Example : the Thue-Morse sequence is a 2-automatic sequence.

The sequence of Thue-Morse on $\{-1, 1\}$ is given by:

$$t_n = \begin{cases} 1 & \text{if the number of 1 in the expansion of } n \text{ in base 2 is even} \\ -1 & \text{otherwise.} \end{cases}$$

$$(t_n)_n = 1 - 1 - 11 - 111 - 1 - 111 - 11 - 1 - 11\ldots$$

$$(t_n)_n = 1 - 1 - 11 - 111 - 1 - 111 - 11 - 1 - 11 \ldots$$

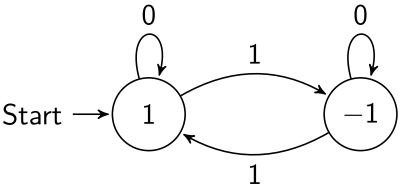

Figure: Automaton which generates the Thue-Morse sequence.

$$(t_n)_n = 1 - 1 - 11 - 111 - 1 - 111 - 11 - 1 - 11\ldots$$

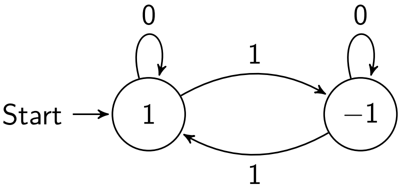

Figure: Automaton which generates the Thue-Morse sequence.

The automatic series $\sum_{n=0}^{+\infty} t_n z^n$ satisfies the following equation:

$$f\left(z^2\right) = \frac{1}{1-z} f(z).$$

Let $x \in [0, 1[$, and let $k \geq 2$ be an integer.

Let $x \in [0, 1[$, and let $k \geq 2$ be an integer.

Let

$$x = \sum_{n=1}^{+\infty} a_n k^{-n}$$

be the expansion of x in base k.

Let $x \in [0, 1[$, and let $k \geq 2$ be an integer.

Let

$$x = \sum_{n=1}^{+\infty} a_n k^{-n}$$

be the expansion of x in base k.

Problem: can we classify the sequence $(a_n)_n$ with respect to a certain notion of complexity ? The complexity can be defined through the notion of automatic sequence.

Let $x \in [0, 1[$, and let $k \geq 2$ be an integer.

Let

$$x = \sum_{n=1}^{+\infty} a_n k^{-n}$$

be the expansion of x in base k.

Problem: can we classify the sequence $(a_n)_n$ with respect to a certain notion of complexity ? The complexity can be defined through the notion of automatic sequence.

Example: if $\sqrt{2} = \sum_{n=0}^{+\infty} a_n 10^{-n}$, is the sequence $(a_n)_n$ automatic ?

Conjecture 2.1 (A. Cobham, 1970)

The expansion of an irrational algebraic number in an integral base cannot be generated by a finite automaton.

Conjecture 2.1 (A. Cobham, 1970)

The expansion of an irrational algebraic number in an integral base cannot be generated by a finite automaton.

The Cobham conjecture is proved in 2007 by B. Adamczewski, Y. Bugeaud, but is also a consequence of the work of P. Philippon, and B. Adamczewski and C. Faverjon about **values of Mahler functions** at algebraic points (2017).

Proposition 2.1 (A. Cobham)

Every k-automatic series is a k-Mahler series.

Proposition 2.1 (A. Cobham)

Every k-automatic series is a k-Mahler series.

Let

$$\sqrt{2} = \sum_{n=0}^{+\infty} a_n 10^{-n}$$

be the decomposition of $\sqrt{2}$ in base 10.

Proposition 2.1 (A. Cobham)

Every k-automatic series is a k-Mahler series.

Let

$$\sqrt{2} = \sum_{n=0}^{+\infty} a_n 10^{-n}$$

be the decomposition of $\sqrt{2}$ in base 10.

Let us assume by **contradiction** that $(a_n)_n$ is automatic.

Proposition 2.1 (A. Cobham)

Every k-automatic series is a k-Mahler series.

Let

$$\sqrt{2} = \sum_{n=0}^{+\infty} a_n 10^{-n}$$

be the decomposition of $\sqrt{2}$ in base 10.

Let us assume by **contradiction** that $(a_n)_n$ is automatic.

Then the series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

is a Mahler function.

Proposition 2.1 (A. Cobham)

Every k-automatic series is a k-Mahler series.

Let

$$\sqrt{2} = \sum_{n=0}^{+\infty} a_n 10^{-n}$$

be the decomposition of $\sqrt{2}$ in base 10.

Let us assume by **contradiction** that $(a_n)_n$ is automatic.

Then the series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

is a Mahler function.

Problem : is $f\left(\frac{1}{10}\right)$ algebraic ?

Proposition 2.1 (A. Cobham)

Every k-automatic series is a k-Mahler series.

Let

$$\sqrt{2} = \sum_{n=0}^{+\infty} a_n 10^{-n}$$

be the decomposition of $\sqrt{2}$ in base 10.

Let us assume by **contradiction** that $(a_n)_n$ is automatic.

Then the series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

is a Mahler function.

Problem : is $f\left(\frac{1}{10}\right)$ algebraic ?

Let us consider a rational fraction $R(z) \in \mathbb{C}(z)$ of degree at least 2, and α a fixed point of $R(z)$. Let us assume that $\alpha = 0$. Then:

Let us consider a rational fraction $R(z) \in \mathbb{C}(z)$ of degree at least 2, and α a fixed point of $R(z)$. Let us assume that $\alpha = 0$. Then:

1. Let $s = R'(0)$. If $s \neq 0$, then, the **Schröder's equation** is:

$$f(sz) = R(f(z)), \qquad \text{(S)}$$

Let us consider a rational fraction $R(z) \in \mathbb{C}(z)$ of degree at least 2, and α a fixed point of $R(z)$. Let us assume that $\alpha = 0$. Then:

1. Let $s = R'(0)$. If $s \neq 0$, then, the **Schröder's equation** is:

$$f(sz) = R(f(z)), \tag{S}$$

2. If $R(z) = \sum_{n=d}^{+\infty} a_n z^n$, where $d \geq 2$, then the **Böttcher's equation** is:

$$f(z^d) = R(f(z)), \tag{B}$$

Let us consider a rational fraction $R(z) \in \mathbb{C}(z)$ of degree at least 2, and α a fixed point of $R(z)$. Let us assume that $\alpha = 0$. Then:

1. Let $s = R'(0)$. If $s \neq 0$, then, the **Schröder's equation** is:

$$f(sz) = R(f(z)), \tag{S}$$

2. If $R(z) = \sum_{n=d}^{+\infty} a_n z^n$, where $d \geq 2$, then the **Böttcher's equation** is:

$$f(z^d) = R(f(z)), \tag{B}$$

3. The **Abel's equation** is:

$$f(R(z)) = f(z) + 1. \tag{A}$$

P.-G. Becker and W. Bergweiler **list** in 1994 **all the differentially algebraic solutions of equations** (S), (B), (A) over $\mathbb{C}(z)$.

P.-G. Becker and W. Bergweiler **list** in 1994 **all the differentially algebraic solutions of equations** (S), (B), (A) over $\mathbb{C}(z)$.

Definition 3.1

A formal power series $f(z)$ with coefficients in the complex plane $\mathbb{C}$ is said to be ***differentially algebraic*** *over $\mathbb{C}(z)$ if there exists a non-zero polynomial $P(z, X_0, \dots, X_n)$ with coefficients in $\mathbb{C}$ such that*

$$P(z, f(z), f'(z), \dots, f^{(n)}(z)) = 0,$$

where $f^{(n)}(z)$ is the n-th derivative of f. Otherwise, we say that f is ***hypertranscendental*** *over $\mathbb{C}(z)$.*

P.-G. Becker and W. Bergweiler **list** in 1994 **all the differentially algebraic solutions of equations** (S), (B), (A) over $\mathbb{C}(z)$.

Definition 3.1

A formal power series $f(z)$ with coefficients in the complex plane $\mathbb{C}$ is said to be ***differentially algebraic*** *over $\mathbb{C}(z)$ if there exists a non-zero polynomial $P(z, X_0, \ldots, X_n)$ with coefficients in $\mathbb{C}$ such that*

$$P(z, f(z), f'(z), \ldots, f^{(n)}(z)) = 0,$$

where $f^{(n)}(z)$ is the n-th derivative of f. Otherwise, we say that f is ***hypertranscendental*** *over $\mathbb{C}(z)$.*

One of the **tools** of the authors: iteration theory/the theory of P. Fatou and G. Julia.

Number field

Number field

$A = \mathbb{Z}$

Number field

$$\begin{array}{c} A = \mathbb{Z} \\ | \\ K = \mathbb{Q} \end{array}$$

Number field

$A = \mathbb{Z}$

$K = \mathbb{Q}$

completion with respect to the classical absolute value over $\mathbb{C}$, written $|.|_\infty$

$R = \mathbb{R}$

Number field

$$
\begin{tikzcd}
A = \mathbb{Z} \arrow[d, dash] & \\
K = \mathbb{Q} \arrow[dr, dash, "\text{completion with respect to the classical absolute value over } \mathbb{C}\text{, written } |.|_\infty"] & \\
& R = \mathbb{R} \arrow[dl, dash, "\text{algebraic closure}"] \\
C = \mathbb{C} &
\end{tikzcd}
$$

Number field

$A = \mathbb{Z}$

$K = \mathbb{Q}$

$< \infty$

$\mathbb{K}$

completion with respect to the classical absolute value over $\mathbb{C}$, written $|.|_\infty$

$R = \mathbb{R}$

algebraic closure

$C = \mathbb{C}$

Number field

$$\begin{tikzcd}
A = \mathbb{Z} \arrow[d, dash] & & \\
K = \mathbb{Q} \arrow[d, dash, "< \infty"'] \arrow[drr, dash, "\text{completion with respect to the classical absolute value over } \mathbb{C}\text{, written } |.|_\infty"] & & \\
\mathbb{K} \arrow[d, dash] & & R = \mathbb{R} \arrow[ddll, dash, "\text{algebraic closure}"] \\
\overline{\mathbb{K}} \arrow[d, dash] & & \\
C = \mathbb{C} & &
\end{tikzcd}$$

Number field

Function field of characteristic $p > 0$

$$
\begin{tikzcd}
A = \mathbb{Z} \arrow[d, dash] & & \\
K = \mathbb{Q} \arrow[d, dash, "<\infty"'] \arrow[ddrr, dash, "\text{completion with respect to the classical absolute value over } \mathbb{C}\text{, written } |.|_\infty"] & & \\
\mathbb{K} \arrow[d, dash] & & \\
\overline{\mathbb{K}} \arrow[d, dash] & & R = \mathbb{R} \arrow[dll, dash, "\text{algebraic closure}"] \\
C = \mathbb{C} & &
\end{tikzcd}
$$

Number field

Function field of characteristic $p > 0$

$$\begin{tikzcd}
A = \mathbb{Z} \arrow[d, dash] & \\
K = \mathbb{Q} \arrow[d, dash, "< \infty"'] \arrow[dr, dash, "\text{completion with respect to the classical absolute value over } \mathbb{C}\text{, written } |.|_\infty"] & \\
\mathbb{K} \arrow[d, dash] & R = \mathbb{R} \arrow[dl, dash] \\
\overline{\mathbb{K}} \arrow[d, dash] & \\
C = \mathbb{C} \arrow[ur, dash, "\text{algebraic closure}"'] &
\end{tikzcd}$$

$$A = \mathbb{F}_q[T], q = p^r$$

Number field	**Function field of characteristic $p > 0$**
$A = \mathbb{Z}$	$A = \mathbb{F}_q[T], q = p^r$
$K = \mathbb{Q}$	$K = \mathbb{F}_q(T)$

$A = \mathbb{Z}$

$|$

$K = \mathbb{Q}$ completion with respect to the classical absolute value over $\mathbb{C}$, written $|.|_\infty$

$< \infty$ $|$

$\mathbb{K}$

$|$

$\overline{\mathbb{K}}$ $R = \mathbb{R}$

$|$ algebraic closure

$C = \mathbb{C}$

Number field

$A = \mathbb{Z}$

$K = \mathbb{Q}$

completion with respect to the classical absolute value over $\mathbb{C}$, written $|.|_\infty$

$< \infty$

$\mathbb{K}$

$R = \mathbb{R}$

$\overline{\mathbb{K}}$

algebraic closure

$C = \mathbb{C}$

Function field of characteristic $p > 0$

$A = \mathbb{F}_q[T], q = p^r$

$K = \mathbb{F}_q(T)$

completion with respect to

$$\left|\frac{P(T)}{Q(T)}\right|_\infty = \left(\frac{1}{q}\right)^{deg_T(Q) - deg_T(P)}$$

$$R = \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$$

Number field

$$\begin{tikzcd} A = \mathbb{Z} \arrow[d, dash] & \\ K = \mathbb{Q} \arrow[d, dash, "< \infty"'] \arrow[dr, dash] & \\ \mathbb{K} \arrow[d, dash] & R = \mathbb{R} \\ \overline{\mathbb{K}} \arrow[d, dash] & \\ C = \mathbb{C} \arrow[ur, dash] & \end{tikzcd}$$

completion with respect to the classical absolute value over $\mathbb{C}$, written $|.|_\infty$

algebraic closure

Function field of characteristic $p > 0$

$$\begin{tikzcd} A = \mathbb{F}_q[T], q = p^r \arrow[d, dash] & \\ K = \mathbb{F}_q(T) \arrow[dr, dash] & \\ & R = \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right) \arrow[d, dash] \\ & \overline{\mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)} \end{tikzcd}$$

completion with respect to $|\frac{P(T)}{Q(T)}|_\infty = \left(\frac{1}{q}\right)^{deg_T(Q) - deg_T(P)}$

algebraic closure

Number field

$A = \mathbb{Z}$

$K = \mathbb{Q}$ — completion with respect to the classical absolute value over $\mathbb{C}$, written $|.|_\infty$ — $R = \mathbb{R}$

$< \infty$

$\mathbb{K}$

$\overline{\mathbb{K}}$

$R = \mathbb{R}$ — algebraic closure — $C = \mathbb{C}$

Function field of characteristic $p > 0$

$A = \mathbb{F}_q[T], q = p^r$

$K = \mathbb{F}_q(T)$ — completion with respect to $|\frac{P(T)}{Q(T)}|_\infty = \left(\frac{1}{q}\right)^{deg_T(Q) - deg_T(P)}$

$R = \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$

algebraic closure

$\overline{\mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)}$

C — completion with respect to $|.|_\infty$

Number field

$A = \mathbb{Z}$

$K = \mathbb{Q}$ completion with respect to the classical absolute value over $\mathbb{C}$, written $|.|_\infty$

$< \infty$

$\mathbb{K}$

$R = \mathbb{R}$

$\overline{\mathbb{K}}$

algebraic closure

$C = \mathbb{C}$

Function field of characteristic $p > 0$

$A = \mathbb{F}_q[T], q = p^r$

$K = \mathbb{F}_q(T)$ completion with respect to $\left|\frac{P(T)}{Q(T)}\right|_\infty = \left(\frac{1}{q}\right)^{deg_T(Q) - deg_T(P)}$

$R = \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$

algebraic closure

$\overline{\mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)}$

C

completion with respect to $|.|_\infty$

Number field

$$A = \mathbb{Z}$$

$$|$$

$$K = \mathbb{Q}$$

completion with respect to the classical absolute value over $\mathbb{C}$, written $|.|_\infty$

$$< \infty \quad |$$

$$\mathbb{K}$$

$$|$$

$$R = \mathbb{R}$$

$$\overline{\mathbb{K}}$$

algebraic closure

$$|$$

$$C = \mathbb{C}$$

Function field of characteristic $p > 0$

$$A = \mathbb{F}_q[T], q = p^r$$

$$|$$

$$K = \mathbb{F}_q(T)$$

completion with respect to $\left|\frac{P(T)}{Q(T)}\right|_\infty = \left(\frac{1}{q}\right)^{deg_T(Q) - deg_T(P)}$

$$< \infty \quad |$$

$$\mathbb{K}$$

$$R = \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$$

$$|$$ algebraic closure

$$\overline{\mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)}$$

$$C$$

completion with respect to $|.|_\infty$

Number field

$$
\begin{array}{l}
A = \mathbb{Z} \\
\quad | \\
K = \mathbb{Q} \\
< \infty \; | \\
\quad \mathbb{K} \\
\quad | \\
\quad \overline{\mathbb{K}} \\
\quad | \\
C = \mathbb{C}
\end{array}
$$

$K = \mathbb{Q}$ — $R = \mathbb{R}$: completion with respect to the classical absolute value over $\mathbb{C}$, written $|.|_\infty$

$R = \mathbb{R}$ — $C = \mathbb{C}$: algebraic closure

Function field of characteristic $p > 0$

$$
\begin{array}{l}
A = \mathbb{F}_q[T], q = p^r \\
\quad | \\
K = \mathbb{F}_q(T) \\
< \infty \; | \\
\quad \mathbb{K} \\
\quad | \\
\quad \overline{\mathbb{K}} \\
\quad | \\
C
\end{array}
$$

$K = \mathbb{F}_q(T)$ — $R = \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$: completion with respect to $\left|\frac{P(T)}{Q(T)}\right|_\infty = \left(\frac{1}{q}\right)^{deg_T(Q)-deg_T(P)}$

$R = \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$ — $\overline{\mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)}$: algebraic closure

$\overline{\mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)}$ — C: completion with respect to $|.|_\infty$

Definition 4.1

*We say that the column vector whose coordinates are the power series $f_1(z), \ldots, f_n(z) \in \mathbb{K}[[z]]$ satisfies a d-**Mahler system** if there exists a matrix $A(z) \in GL_n(\mathbb{K}(z))$ such that*

$$\begin{pmatrix} f_1(z^d) \\ \vdots \\ f_n(z^d) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix}. \tag{2}$$

Motivations

1. Find an other general setting in which we can get results of transcendence and algebraic independence of numbers.

Motivations

1. Find an other general setting in which we can get results of transcendence and algebraic independence of numbers.
2. Specificity of positive characteristic: values of Mahler functions- as the analogue of π, or more generally some periods of Drinfeld modules- are values of Mahler functions at some algebraic points.

$$\pi_q = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{T^{q^i}}\right)^{-1} \in \mathbb{F}_q((1/T))$$

$$\pi_q = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{T^{q^i}}\right)^{-1} \in \mathbb{F}_q((1/T))$$

Question: is the number π_q transcendental over $\overline{\mathbb{F}_q(T)}$?

$$\pi_q = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{T^{q^i}}\right)^{-1} \in \mathbb{F}_q((1/T))$$

Question: is the number π_q transcendental over $\overline{\mathbb{F}_q(T)}$?

Let us set

$$f(z) = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{z^{q^i}}\right)^{-1} \in \mathbb{F}_q(T)\{1/z\}.$$

$$\pi_q = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{T^{q^i}}\right)^{-1} \in \mathbb{F}_q((1/T))$$

Question: is the number π_q transcendental over $\overline{\mathbb{F}_q(T)}$?

Let us set

$$f(z) = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{z^{q^i}}\right)^{-1} \in \mathbb{F}_q(T)\{1/z\}.$$

We get:

$$\pi_q = f(T),$$

$$\pi_q = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{T^{q^i}}\right)^{-1} \in \mathbb{F}_q((1/T))$$

Question: is the number π_q transcendental over $\overline{\mathbb{F}_q(T)}$?

Let us set

$$f(z) = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{z^{q^i}}\right)^{-1} \in \mathbb{F}_q(T)\{1/z\}.$$

We get:

$$\pi_q = f(T),$$

$$f(z^q) = \left(1 - \frac{T}{z^q}\right) f(z).$$

$$\pi_q = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{T^{q^i}}\right)^{-1} \in \mathbb{F}_q((1/T))$$

Question: is the number π_q transcendental over $\overline{\mathbb{F}_q(T)}$?

Let us set

$$f(z) = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{z^{q^i}}\right)^{-1} \in \mathbb{F}_q(T)\{1/z\}.$$

We get:

$$\pi_q = f(T),$$

$$f(z^q) = \left(1 - \frac{T}{z^q}\right) f(z).$$

Theorem 4.1 (F.)

Let $n \geq 1$, $d \geq 2$ *be two integers and* $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$ *be functions satisfying d-Mahler System* (2). *Let* $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$, *be a regular number with respect to System* (2). *Then*

$$\mathrm{trdeg}_{\overline{\mathbb{K}}}\{f_1(\alpha), \ldots, f_n(\alpha)\} = \mathrm{trdeg}_{\overline{\mathbb{K}}(z)}\{f_1(z), \ldots, f_n(z)\}.$$

Theorem 4.1 (F.)

Let $n \geq 1$, $d \geq 2$ be two integers and $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$ be functions satisfying d-Mahler System (2). Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$, be a regular number with respect to System (2). Then

$$\operatorname{trdeg}_{\overline{\mathbb{K}}}\{f_1(\alpha), \ldots, f_n(\alpha)\} = \operatorname{trdeg}_{\overline{\mathbb{K}}(z)}\{f_1(z), \ldots, f_n(z)\}.$$

Definition 4.2

We say that a number $\alpha \in \mathbb{K}$ is ***regular*** *with respect to System (2) if for all integer $k \geq 0$, the number α^{d^k} is neither a pole of the matrix $A(z)$ nor a pole of the matrix $A^{-1}(z)$.*

Theorem 4.1 (F.)

Let $n \geq 1$, $d \geq 2$ be two integers and $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$ be functions satisfying d-Mahler System (2). Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$, be a regular number with respect to System (2). Then

$$\operatorname{trdeg}_{\overline{\mathbb{K}}}\{f_1(\alpha), \ldots, f_n(\alpha)\} = \operatorname{trdeg}_{\overline{\mathbb{K}}(z)}\{f_1(z), \ldots, f_n(z)\}.$$

Definition 4.2

*We say that a number $\alpha \in \mathbb{K}$ is **regular** with respect to System (2) if for all integer $k \geq 0$, the number α^{d^k} is neither a pole of the matrix $A(z)$ nor a pole of the matrix $A^{-1}(z)$.*

When $\mathbb{K}$ is a number field, this result is proved by Ku. Nishioka in 1991.

Limits:

1. We deal with algebraic independence of functions, which is an open question in general for Mahler functions. Linear independence of such functions is easier (Algorithm of B. Adamczewski and C. Faverjon in characteristic zero).

Limits:

1. We deal with algebraic independence of functions, which is an open question in general for Mahler functions. Linear independence of such functions is easier (Algorithm of B. Adamczewski and C. Faverjon in characteristic zero).
2. If the Mahler function $f(z)$ is transcendental, we cannot conclude that $f(\alpha)$ is too.

Theorem 4.2 (F.)

We continue with the assumptions of Theorem 4.1.

Theorem 4.2 (F.)

We continue with the assumptions of Theorem 4.1. Let us assume further that the extension $\overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\mathbb{K}}(z)$.

Theorem 4.2 (F.)

We continue with the assumptions of Theorem 4.1. Let us assume further that the extension $\overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\mathbb{K}}(z)$.
Then, for every polynomial $P(X_1, \ldots, X_n) \in \overline{\mathbb{K}}[X_1, \ldots, X_n]$ homogeneous in $X_1, \ldots, X_n$ such that

$$P(f_1(\alpha), \ldots, f_n(\alpha)) = 0,$$

Theorem 4.2 (F.)

We continue with the assumptions of Theorem 4.1. Let us assume further that the extension $\overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\mathbb{K}}(z)$.
Then, for every polynomial $P(X_1, \ldots, X_n) \in \overline{\mathbb{K}}[X_1, \ldots, X_n]$ homogeneous in $X_1, \ldots, X_n$ such that

$$P(f_1(\alpha), \ldots, f_n(\alpha)) = 0,$$

there exists a polynomial $Q(z, X_1, \ldots, X_n) \in \overline{\mathbb{K}}[z][X_1, \ldots, X_n]$ homogeneous in $X_1, \ldots, X_n$ such that

$$Q(z, f_1(z), \ldots, f_n(z)) = 0,$$

and

Theorem 4.2 (F.)

We continue with the assumptions of Theorem 4.1. Let us assume further that the extension $\overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\mathbb{K}}(z)$.
Then, for every polynomial $P(X_1, \ldots, X_n) \in \overline{\mathbb{K}}[X_1, \ldots, X_n]$ homogeneous in $X_1, \ldots, X_n$ such that

$$P(f_1(\alpha), \ldots, f_n(\alpha)) = 0,$$

there exists a polynomial $Q(z, X_1, \ldots, X_n) \in \overline{\mathbb{K}}[z][X_1, \ldots, X_n]$ homogeneous in $X_1, \ldots, X_n$ such that

$$Q(z, f_1(z), \ldots, f_n(z)) = 0,$$

and

$$Q(\alpha, X_1, \ldots, X_n) = P(X_1, \ldots, X_n).$$

Corollary 4.1

We continue with the assumptions of Theorem 4.2. If the functions $f_1(z), \ldots, f_n(z)$ are linearly independent over $\overline{\mathbb{K}}(z)$, then, the numbers $f_1(\alpha), \ldots, f_n(\alpha)$ are linearly independent over $\overline{\mathbb{K}}$.

Corollary 4.2

Let $f(z) \in \mathbb{K}\{z\}$ *be a d-Mahler transcendental function over* $\mathbb{K}(z)$.

Corollary 4.2

Let $f(z) \in \mathbb{K}\{z\}$ be a d-Mahler transcendental function over $\mathbb{K}(z)$.

Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ such that α is in the disc of convergence of $f(z)$.

Corollary 4.2

Let $f(z) \in \mathbb{K}\{z\}$ be a d-Mahler transcendental function over $\mathbb{K}(z)$.

Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ such that α is in the disc of convergence of $f(z)$.

Let us assume that the extension $\overline{\mathbb{K}}(z)(f(z))_{\sigma_d}$ is regular over $\overline{\mathbb{K}}(z)$.

Corollary 4.2

Let $f(z) \in \mathbb{K}\{z\}$ be a d-Mahler transcendental function over $\mathbb{K}(z)$.

Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ such that α is in the disc of convergence of $f(z)$.

Let us assume that the extension $\overline{\mathbb{K}}(z)(f(z))_{\sigma_d}$ is regular over $\overline{\mathbb{K}}(z)$.

Then, we have the following.

Corollary 4.2

Let $f(z) \in \mathbb{K}\{z\}$ be a d-Mahler transcendental function over $\mathbb{K}(z)$.

Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ such that α is in the disc of convergence of $f(z)$.

Let us assume that the extension $\overline{\mathbb{K}}(z)(f(z))_{\sigma_d}$ is regular over $\overline{\mathbb{K}}(z)$.

Then, we have the following.

1. *The number $f(\alpha)$ is either transcendental or in $\mathbb{K}(\alpha)$.*

Corollary 4.2

Let $f(z) \in \mathbb{K}\{z\}$ be a d-Mahler transcendental function over $\mathbb{K}(z)$.

Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ such that α is in the disc of convergence of $f(z)$.

Let us assume that the extension $\overline{\mathbb{K}}(z)(f(z))_{\sigma_d}$ is regular over $\overline{\mathbb{K}}(z)$.

Then, we have the following.

1. *The number $f(\alpha)$ is either transcendental or in $\mathbb{K}(\alpha)$.*
2. *If α is a regular number with respect to the minimal d-Mahler System satisfied by $f(z)$, then $f(\alpha)$ is transcendental over $\overline{\mathbb{K}}$.*

When $\mathbb{K}$ is a number field, the analogue of the theorem and corollaries are proved by B. Adamczewski and C. Faverjon, as a consequence of the work of P. Philippon (2017).

When $\mathbb{K}$ is a number field, the analogue of the theorem and corollaries are proved by B. Adamczewski and C. Faverjon, as a consequence of the work of P. Philippon (2017).

Cobham conjecture. Let

$$\sqrt{2} = \sum_{n=0}^{+\infty} a_n 10^{-n}$$

be the decomposition of $\sqrt{2}$ in base 10.

Let us assume by **contradiction** that $(a_n)_n$ is automatic.

Then the series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

is a Mahler function.

When $\mathbb{K}$ is a number field, the analogue of the theorem and corollaries are proved by B. Adamczewski and C. Faverjon, as a consequence of the work of P. Philippon (2017).

Cobham conjecture. Let

$$\sqrt{2} = \sum_{n=0}^{+\infty} a_n 10^{-n}$$

be the decomposition of $\sqrt{2}$ in base 10.

Let us assume by **contradiction** that $(a_n)_n$ is automatic.

Then the series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

is a Mahler function.

Then $f\left(\frac{1}{10}\right)$ is **transcendental or rational**: **contradiction**.

Definition 4.3

If $f_1(z), \ldots, f_n(z)$ are d-Mahler functions over $\mathbb{K}$, we say that the Mahler extension $\mathscr{E} = \overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is ***regular*** *over $\overline{\mathbb{K}}(z)$ if every algebraic element of $\mathscr{E}$ belongs to $\overline{\mathbb{K}}(z)$.*

Definition 4.3

If $f_1(z), \ldots, f_n(z)$ are d-Mahler functions over $\mathbb{K}$, we say that the Mahler extension $\mathscr{E} = \overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is ***regular*** *over $\overline{\mathbb{K}}(z)$ if every algebraic element of $\mathscr{E}$ belongs to $\overline{\mathbb{K}}(z)$.*

1. In characteristic zero, every Mahler extension is regular. Because a Mahler function is either transcendental or rational.

Definition 4.3

*If $f_1(z), \ldots, f_n(z)$ are d-Mahler functions over $\mathbb{K}$, we say that the Mahler extension $\mathscr{E} = \overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is **regular** over $\overline{\mathbb{K}}(z)$ if every algebraic element of $\mathscr{E}$ belongs to $\overline{\mathbb{K}}(z)$.*

1. In characteristic zero, every Mahler extension is regular. Because a Mahler function is either transcendental or rational.
2. In positive characteristic, this is no longer true.

Definition 4.3

If $f_1(z), \ldots, f_n(z)$ are d-Mahler functions over $\mathbb{K}$, we say that the Mahler extension $\mathscr{E} = \overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is ***regular*** *over $\overline{\mathbb{K}}(z)$ if every algebraic element of $\mathscr{E}$ belongs to $\overline{\mathbb{K}}(z)$.*

1. In characteristic zero, every Mahler extension is regular. Because a Mahler function is either transcendental or rational.
2. In positive characteristic, this is no longer true.

Example:

Definition 4.3

If $f_1(z), \ldots, f_n(z)$ are d-Mahler functions over $\mathbb{K}$, we say that the Mahler extension $\mathscr{E} = \overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is ***regular*** *over $\overline{\mathbb{K}}(z)$ if every algebraic element of $\mathscr{E}$ belongs to $\overline{\mathbb{K}}(z)$.*

1. In characteristic zero, every Mahler extension is regular. Because a Mahler function is either transcendental or rational.
2. In positive characteristic, this is no longer true.

Example:

Let consider the p-Mahler system:

$$\begin{pmatrix} f_1(z^p) \\ f_2(z^p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}.$$

Definition 4.3

*If $f_1(z), \ldots, f_n(z)$ are d-Mahler functions over $\mathbb{K}$, we say that the Mahler extension $\mathscr{E} = \overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is **regular** over $\overline{\mathbb{K}}(z)$ if every algebraic element of $\mathscr{E}$ belongs to $\overline{\mathbb{K}}(z)$.*

1. In characteristic zero, every Mahler extension is regular. Because a Mahler function is either transcendental or rational.
2. In positive characteristic, this is no longer true.

Example:

Let consider the p-Mahler system:

$$\begin{pmatrix} f_1(z^p) \\ f_2(z^p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}.$$

A solution of this system is given by

$$f_1(z) = 1, f_2(z) = \sum_{n=0}^{+\infty} z^{p^n}.$$

A solution of this system is given by

$$f_1(z) = 1, f_2(z) = \sum_{n=0}^{+\infty} z^{p^n}.$$

The function $f_2(z)$ is **algebraic** because $f_2(z)^p = f_2(z^p) = f_2(z) - z$.

A solution of this system is given by

$$f_1(z) = 1, f_2(z) = \sum_{n=0}^{+\infty} z^{p^n}.$$

The function $f_2(z)$ is **algebraic** because $f_2(z)^p = f_2(z^p) = f_2(z) - z$.

But the function $f_2(z)$ is **not rational**.

A solution of this system is given by

$$f_1(z) = 1, f_2(z) = \sum_{n=0}^{+\infty} z^{p^n}.$$

The function $f_2(z)$ is **algebraic** because
$f_2(z)^p = f_2(z^p) = f_2(z) - z$.

But the function $f_2(z)$ is **not rational**.

It follows that the extension $\mathscr{E} = \overline{\mathbb{K}}(z)(f_1(z), f_2(z))$ is **not regular** over $\overline{\mathbb{K}}(z)$.

Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ and $\lambda = f_2(\alpha) \in \overline{\mathbb{K}}$.

Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ and $\lambda = f_2(\alpha) \in \overline{\mathbb{K}}$.

Then, $\lambda f_1(\alpha) - f_2(\alpha) = 0$ is a non-trivial **linear relation** between $f_1(\alpha)$ and $f_2(\alpha)$ over $\overline{\mathbb{K}}$.

Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ and $\lambda = f_2(\alpha) \in \overline{\mathbb{K}}$.

Then, $\lambda f_1(\alpha) - f_2(\alpha) = 0$ is a non-trivial **linear relation** between $f_1(\alpha)$ and $f_2(\alpha)$ over $\overline{\mathbb{K}}$.

However, there is **no non-trivial linear relation** between the function $f_1(z)$ and $f_2(z)$ over $\overline{\mathbb{K}}(z)$, because $f_2(z)$ is not rational.

Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ and $\lambda = f_2(\alpha) \in \overline{\mathbb{K}}$.

Then, $\lambda f_1(\alpha) - f_2(\alpha) = 0$ is a non-trivial **linear relation** between $f_1(\alpha)$ and $f_2(\alpha)$ over $\overline{\mathbb{K}}$.

However, there is **no non-trivial linear relation** between the function $f_1(z)$ and $f_2(z)$ over $\overline{\mathbb{K}}(z)$, because $f_2(z)$ is not rational.

Hence, the conclusion of Theorem 4.2 **does not hold** in this case.

1. We proved that the condition of regularity of the Mahler extension is necessary for the conclusion of Theorem 4.2 to hold.

1. We proved that the condition of regularity of the Mahler extension is necessary for the conclusion of Theorem 4.2 to hold.
2. We proved that if $p \nmid d$, then, the extension $\mathscr{E}$ is regular.

1. We proved that the condition of regularity of the Mahler extension is necessary for the conclusion of Theorem 4.2 to hold.
2. We proved that if $p \nmid d$, then, the extension $\mathscr{E}$ is regular.
3. Concrete examples usually are in the case $p \mid d$.

1. We obtained results of **functional transcendence and algebraic independence** for inhomogeneous Mahler equations of **order** 1. By a generalisation of results of L. Denis.

1. We obtained results of **functional transcendence and algebraic independence** for inhomogeneous Mahler equations of **order** 1. By a generalisation of results of L. Denis.
2. For general Mahler equations in characteristic zero: **Galois theory**. Work of J. Roques for Mahler equations of order 2.

Thank you for your attention !