Algebraic relations between values of Mahler functions in positive characteristic

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Definition 1.1

A number field $\mathbb{K}$ is a finite extension of $\mathbb{Q}$. That is a field which contains $\mathbb{Q}$ and of finite dimension, as a $\mathbb{Q}$-vector space.
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2. We say that $x \in \mathbb{C}$ is algebraic over $\mathbb{K}$ if there exists a non-zero polynomial $P(X) \in \mathbb{K}[X]$ such that $P(x) = 0$. Otherwise, we say that $x$ is transcendental over $\mathbb{K}$.
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3. We let $\overline{\mathbb{K}}$ be the set all the **algebraic** complex numbers over $\mathbb{K}$.
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3. We let $\overline{\mathbb{K}}$ be the set all the **algebraic** complex numbers over $\mathbb{K}$.

4. We say that $x_1, \ldots, x_n \in \mathbb{C}$ are **algebraically dependent** over $\mathbb{K}$ if there exists a non-zero polynomial $P(X_1, \ldots, X_n) \in \mathbb{K}[X_1, \ldots, X_n]$ such that $P(x_1, \ldots, x_n) = 0$. 

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Definition 1.2

Let $d \geq 2$ be an integer. Let $\mathbb{K}$ be a number field. Let $f(z) \in \mathbb{K}\{z\}$ be a convergent power series in a neighbourhood of the origin, with coefficients in $\mathbb{K}$. 
Definition 1.2

Let $d \geq 2$ be an integer. Let $\mathbb{K}$ be a number field. Let $f(z) \in \mathbb{K}\{z\}$ be a convergent power series in a neighbourhood of the origin, with coefficients in $\mathbb{K}$.

Then, $f(z)$ is $d$-Mahler over $\mathbb{K}(z)$ if there exist polynomials $P_0(z), \ldots, P_n(z) \in \mathbb{K}[z]$, $P_n(z) \neq 0$, such that:

$$P_0(z)f(z) + P_1(z)f(z^d) + \cdots + P_n(z)f(z^{d^n}) = 0.$$
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Example: $f(z) = \sum_{n=0}^{+\infty} z^{2^n}$. 
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Example: $f(z) = \sum_{n=0}^{+\infty} z^{2^n}$.

$$f \left( z^2 \right) = f(z) - z.$$
Definition 1.3

We say that the column vector whose coordinates are the power series $f_1(z), \ldots, f_n(z) \in \mathbb{K}[[z]]$ satisfies a $d$-Mahler system if there exists a matrix $A(z) \in \text{GL}_n(\mathbb{K}(z))$ such that

$$
\begin{pmatrix}
 f_1(z^d) \\
 \vdots \\
 f_n(z^d)
\end{pmatrix}
 = A(z)
\begin{pmatrix}
 f_1(z) \\
 \vdots \\
 f_n(z)
\end{pmatrix}.
$$

(1)
Problem
Problem

Let $f(z) \in \mathbb{K}\{z\}$, and $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$ be $d$-Mahler functions, and let $\alpha \in \overline{\mathbb{K}}$. 
Problem

Let $f(z) \in \mathbb{K}\{z\}$, and $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$ be $d$-Mahler functions, and let $\alpha \in \overline{\mathbb{K}}$.

We are interested in results of transcendence of $f(\alpha)$ or algebraic independence of $f_1(\alpha), \ldots, f_n(\alpha)$ over $\overline{\mathbb{K}}(z)$. 

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Definition 1

Let $k \geq 2$ be an integer. A sequence $(a_n)_n$ is *k-automatic* if it is generated by a finite $k$-automaton: a machine which reads, for every integer $n$, the sequence of the expansion of $n$ in base $k$ and returns the value $a_n$. 

Example: the Thue-Morse sequence is a 2-automatic sequence. The sequence of Thue-Morse on $\{-1, 1\}$ is given by:

$$ t_n = \begin{cases} -1 & \text{if the number of 1 in the expansion of } n \text{ in base 2 is even} \\ -1 & \text{otherwise.} \end{cases} $$
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**Example**: the Thue-Morse sequence is a 2-automatic sequence.

The sequence of Thue-Morse on \( \{-1, 1\} \) is given by:

\[
t_n = \begin{cases} 
1 & \text{if the number of 1 in the expansion of } n \text{ in base 2 is even} \\
-1 & \text{otherwise.}
\end{cases}
\]
\[(t_n)_n = 1 - 1 - 11 - 111 - 1 - 111 - 11 - 1 - 11...\]
\((t_n)_n = 1 - 1 - 11 - 111 - 1 - 111 - 11 - 1 - 11...\)

**Figure:** Automaton which generates the Thue-Morse sequence.
\((t_n)_n = 1 - 1 - 11 - 111 - 1 - 111 - 11 - 1 - 11\ldots\)

![Automaton which generates the Thue-Morse sequence.](image)

**Figure**: Automaton which generates the Thue-Morse sequence.

The automatic series \(\sum_{n=0}^{+\infty} t_n z^n\) satisfies the following equation:

\[
f \left( z^2 \right) = \frac{1}{1 - z} f(z).
\]
Let $x \in [0, 1]$, and let $k \geq 2$ be an integer.
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Let

$$x = \sum_{n=1}^{+\infty} a_n k^{-n}$$

be the expansion of $x$ in base $k$. 

---

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\]

be the expansion of \( x \) in base \( k \).

**Problem**: can we classify the sequence \((a_n)_n\) with respect to a certain notion of complexity? The complexity can be defined through the notion of automatic sequence.
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Let

$$x = \sum_{n=1}^{+\infty} a_n k^{-n}$$

be the expansion of $x$ in base $k$.

**Problem**: can we classify the sequence $(a_n)_n$ with respect to a certain notion of complexity? The complexity can be defined through the notion of automatic sequence.

**Example**: if $\sqrt{2} = \sum_{n=0}^{+\infty} a_n 10^{-n}$, is the sequence $(a_n)_n$ automatic?
Conjecture 2.1 (A. Cobham, 1970)

The expansion of an irrational algebraic number in an integral base cannot be generated by a finite automaton.
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The expansion of an irrational algebraic number in an integral base cannot be generated by a finite automaton.

The Cobham conjecture is proved in 2007 by B. Adamczewski, Y. Bugeaud, but is also a consequence of the work of P. Philippon, and B. Adamczewski and C. Faverjon about values of Mahler functions at algebraic points (2017).
Proposition 2.1 (A. Cobham)

Every \( k \)-automatic series is a \( k \)-Mahler series.
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Every $k$-automatic series is a $k$-Mahler series.

Let

$$\sqrt{2} = \sum_{n=0}^{+\infty} a_n 10^{-n}$$

be the decomposition of $\sqrt{2}$ in base 10.
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Let us assume by contradiction that $(a_n)_n$ is automatic.
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Then the series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

is a Mahler function.
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*Every k-automatic series is a k-Mahler series.*

Let

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be the decomposition of \( \sqrt{2} \) in base 10.

Let us assume by *contradiction* that \((a_n)_n\) is automatic.

Then the series

\[ f(z) = \sum_{n=0}^{+\infty} a_n z^n \]

is a Mahler function.

**Problem**: is \( f \left( \frac{1}{10} \right) \) algebraic?
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be the decomposition of $\sqrt{2}$ in base 10.

Let us assume by contradiction that $(a_n)_n$ is automatic.

Then the series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

is a Mahler function.

Problem: is $f\left(\frac{1}{10}\right)$ algebraic?
Let us consider a rational fraction $R(z) \in \mathbb{C}(z)$ of degree at least 2, and $\alpha$ a fixed point of $R(z)$. Let us assume that $\alpha = 0$. Then:
Let us consider a rational fraction \( R(z) \in \mathbb{C}(z) \) of degree at least 2, and \( \alpha \) a fixed point of \( R(z) \). Let us assume that \( \alpha = 0 \). Then:

1. Let \( s = R'(0) \). If \( s \neq 0 \), then, the **Schröder’s equation** is:

   \[
   f(sz) = R(f(z)),
   \]

   (S)
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1. Let $s = R'(0)$. If $s \neq 0$, then, the **Schröder’s equation** is:

   $$ f(sz) = R(f(z)), \quad (S) $$

2. If $R(z) = \sum_{n=d}^{+\infty} a_n z^n$, where $d \geq 2$, then the **Böttcher’s equation** is:

   $$ f(z^d) = R(f(z)), \quad (B) $$
Let us consider a rational fraction $R(z) \in \mathbb{C}(z)$ of degree at least 2, and $\alpha$ a fixed point of $R(z)$. Let us assume that $\alpha = 0$. Then:

1. Let $s = R'(0)$. If $s \neq 0$, then, the **Schröder’s equation** is:

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2. If $R(z) = \sum_{n=d}^{+\infty} a_n z^n$, where $d \geq 2$, then the **Böttcher’s equation** is:

   $$ f(z^d) = R(f(z)), \quad (B) $$

3. The **Abel’s equation** is:

   $$ f(R(z)) = f(z) + 1. \quad (A) $$
P.-G. Becker and W. Bergweiler list in 1994 all the differentially algebraic solutions of equations (S), (B), (A) over $\mathbb{C}(z)$. 
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**Definition 3.1**

A formal power series \( f(z) \) with coefficients in the complex plane \( \mathbb{C} \) is said to be **differentially algebraic** over \( \mathbb{C}(z) \) if there exists a non-zero polynomial \( P(z, X_0, \ldots, X_n) \) with coefficients in \( \mathbb{C} \) such that

\[
P(z, f(z), f'(z), \ldots, f^{(n)}(z)) = 0,
\]

where \( f^{(n)}(z) \) is the \( n \)-th derivative of \( f \). Otherwise, we say that \( f \) is **hypertranscendental** over \( \mathbb{C}(z) \).
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where $f^{(n)}(z)$ is the $n$-th derivative of $f$. Otherwise, we say that $f$ is **hypertranscendental** over $\mathbb{C}(z)$.

One of the **tools** of the authors: iteration theory/the theory of P. Fatou and G. Julia.
Number field
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\[ A = \mathbb{Z} \]
Number field

\[ A = \mathbb{Z} \]
\[ | \]
\[ K = \mathbb{Q} \]
Number field

\[ A = \mathbb{Z} \]

\[ K = \mathbb{Q} \quad \text{completion with respect to the classical absolute value over } \mathbb{C}, \text{ written } |.|_\infty \]

\[ R = \mathbb{R} \]
Number field

\[ A = \mathbb{Z} \]
\[ K = \mathbb{Q} \] completion with respect to the classical absolute value over \( \mathbb{C} \), written \(|.|_\infty\)
\[ R = \mathbb{R} \]
\[ C = \mathbb{C} \] algebraic closure
Number field

\[ A = \mathbb{Z} \]
\[ K = \mathbb{Q} \]
\[ K_{<\infty} \]
\[ R = \mathbb{R} \]

completion with respect to the classical absolute value over \( \mathbb{C} \), written \( | \cdot |_\infty \)

algebraic closure

\[ C = \mathbb{C} \]
Number field

\[ A = \mathbb{Z} \]
\[ K = \mathbb{Q} \] completion with respect to the classical absolute value over \( \mathbb{C} \), written \( | \cdot |_\infty \)
\[ < \infty \]
\[ \mathbb{K} \]
\[ R = \mathbb{R} \]
\[ \overline{K} \]
\[ \mathbb{C} = \mathbb{C} \] algebraic closure
**Number field**

$A = \mathbb{Z}$

$K = \mathbb{Q}$ completion with respect to the classical absolute value over $\mathbb{C}$, written $| \cdot |_\infty$

$\mathbb{C} = \mathbb{C}$

**Function field of characteristic $p > 0$**

$K < \infty$ completion with respect to $\mathbb{C}$

$R = \mathbb{R}$

$\overline{K}$ algebraic closure
### Number field

\[ A = \mathbb{Z} \]

\[ K = \mathbb{Q} \]

completion with respect to the classical absolute value over \( \mathbb{C} \), written \( | \cdot |_\infty \)

\[ \mathcal{K} \]

\[ R = \mathbb{R} \]

algebraic closure

\[ C = \mathbb{C} \]

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### Function field of characteristic \( p > 0 \)

\[ A = \mathbb{F}_q[T], q = p^r \]

\[ \mathcal{K} \]

\[ R = \mathbb{R} \]

algebraic closure

\[ C = \mathbb{C} \]
Number field

\[ A = \mathbb{Z} \]
\[ K = \mathbb{Q} \]
\[ C = \mathbb{C} \]

Function field of characteristic \( p > 0 \)

\[ A = \mathbb{F}_q[T], q = p^r \]
\[ K = \mathbb{F}_q(T) \]

- \( K \) completion with respect to the classical absolute value over \( \mathbb{C} \), written \( | \cdot |_{\infty} \)
- \( R = \mathbb{R} \)
- \( \overline{K} \) algebraic closure
**Number field**

\[
\begin{align*}
A &= \mathbb{Z} \\
K &= \mathbb{Q} \quad \text{completion with respect to the classical absolute value over } \mathbb{C}, \text{ written } | \cdot |_\infty \\
\mathcal{C} &= \mathbb{C}
\end{align*}
\]

**Function field of characteristic } p > 0**

\[
\begin{align*}
A &= \mathbb{F}_q[T], \quad q = p^r \\
K &= \mathbb{F}_q(T) \quad \text{completion with respect to } \frac{P(T)}{Q(T)}|_\infty = \left( \frac{1}{q} \right)^{\deg(T) - \deg(P)} \\
\mathcal{R} &= \mathbb{F}_q\left( \left( \frac{1}{T} \right) \right)
\end{align*}
\]
Number field

\[ A = \mathbb{Z} \]
\[ K = \mathbb{Q} \text{ completion with respect to the classical absolute value over } \mathbb{C}, \text{ written } |.|_\infty \]
\[ \mathbb{K} \]
\[ \mathbb{K} \text{ algebraic closure} \]
\[ C = \mathbb{C} \]

Function field of characteristic \( p > 0 \)

\[ A = \mathbb{F}_q[T], \quad q = p^r \]
\[ K = \mathbb{F}_q(T) \text{ completion with respect to } |P(T)Q(T)|_\infty = \left(\frac{1}{q}\right)^{\deg T(Q) - \deg T(P)} \]
\[ R = \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right) \text{ algebraic closure} \]
\[ \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right) \]
Merah functions
Links with the theory of finite automata
Links with dynamical systems
Results on algebraic independence of Mahler values
Algebraic independence of Mahler functions

Motivations
First result
Improvement of the first result
Regular extensions

Number field

\[ A = \mathbb{Z} \]
\[ K = \mathbb{Q} \text{ completion with respect to the classical absolute value over } \mathbb{C}, \text{ written } |.|_\infty \]
\[ < \infty \]
\[ K \]
\[ R = \mathbb{R} \]
\[ \overline{K} \]
\[ C = \mathbb{C} \]

Function field of characteristic \( p > 0 \)

\[ A = \mathbb{F}_q[T], q = p^r \]
\[ K = \mathbb{F}_q(T) \text{ completion with respect to } \frac{|P(T)|}{|Q(T)|}_\infty = \left( \frac{1}{q} \right)^{\deg_T(Q) - \deg_T(P)} \]
\[ R = \mathbb{F}_q \left( \left( \frac{1}{T} \right) \right) \text{ algebraic closure } \]
\[ C = \mathbb{C} \text{ completion with respect to } |.|_\infty \]
**Number field**

\[
A = \mathbb{Z} \\
K = \mathbb{Q} \\
< \infty \\
K \\
R = \mathbb{R} \\
\mathbb{C} = \mathbb{C}
\]

Completion with respect to the classical absolute value over \( \mathbb{C} \), written \( | \cdot |_\infty \).

**Function field of characteristic** \( p > 0 \)

\[
A = \mathbb{F}_q[T], q = p^r \\
K = \mathbb{F}_q(T) \\
\text{completion with respect to} \\
| \frac{P(T)}{Q(T)} |_\infty = \left( \frac{1}{q} \right)^{\deg_T(Q) - \deg_T(P)} \\
R = \mathbb{F}_q \left( \left( \frac{1}{T} \right) \right) \\
\text{algebraic closure} \\
\mathbb{F}_q \left( \left( \frac{1}{T} \right) \right) \\
\mathbb{C} = \mathbb{C} \\
\text{completion with respect to } | \cdot |_\infty
\]
Number field

\[ A = \mathbb{Z} \]
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\[ \mathbb{C} = \mathbb{C} \]

Function field of characteristic \( p > 0 \)

\[ A = \mathbb{F}_q[T], q = p^r \]
\[ K = \mathbb{F}_q(T) \] completion with respect to \( \frac{P(T)}{Q(T)} \)
\[ R = \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right) \]
\[ \overline{K} \]
\[ \text{algebraic closure} \]
\[ \mathbb{C} = \mathbb{C} \] completion with respect to \(| \cdot |_\infty\)
Number field

\[ A = \mathbb{Z} \]
\[ K = \mathbb{Q} \]
\[ C = \mathbb{C} \]
\[ \text{completion with respect to the classical absolute value over } \mathbb{C}, \text{ written } |.|_{\infty} \]
\[ R = \mathbb{R} \]
\[ \text{algebraic closure} \]

Function field of characteristic \( p > 0 \)

\[ A = \mathbb{F}_q[T], q = p^r \]
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\[ R = \mathbb{F}_q\left(\frac{1}{T}\right) \]
\[ \text{algebraic closure} \]
\[ \mathbb{F}_q\left(\frac{1}{T}\right) \]
\[ \text{completion with respect to } |.|_{\infty} \]
Definition 4.1

We say that the column vector whose coordinates are the power series $f_1(z), \ldots, f_n(z) \in K[[z]]$ satisfies a $d$-Mahler system if there exists a matrix $A(z) \in GL_n(K(z))$ such that

$$
\begin{bmatrix}
  f_1(z^d) \\
  \vdots \\
  f_n(z^d)
\end{bmatrix}
= A(z)
\begin{bmatrix}
  f_1(z) \\
  \vdots \\
  f_n(z)
\end{bmatrix}.
$$

(2)
Motivations

1. Find an other general setting in which we can get results of transcendence and algebraic independence of numbers.
Motivations

1. Find an other general setting in which we can get results of transcendence and algebraic independence of numbers.

2. Specificity of positive characteristic: values of Mahler functions—as the analogue of \( \pi \), or more generally some periods of Drinfeld modules—are values of Mahler functions at some algebraic points.
\[ \pi_q = \prod_{i=1}^{+\infty} \left( 1 - \frac{T}{Tq^i} \right)^{-1} \in \mathbb{F}_q((1/T)) \]
\[ \pi_q = \prod_{i=1}^{+\infty} \left( 1 - \frac{T}{Tq^i} \right)^{-1} \in \mathbb{F}_q(\frac{1}{T}) \]

**Question:** is the number \( \pi_q \) transcendental over \( \overline{\mathbb{F}_q(T)} \) ?
\[ \pi_q = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{Tq^i}\right)^{-1} \in \mathbb{F}_q((1/T)) \]

**Question:** is the number \( \pi_q \) transcendental over \( \mathbb{F}_q(T) \)?

Let us set

\[ f(z) = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{zq^i}\right)^{-1} \in \mathbb{F}_q(T)\{1/z\}. \]
\[ \pi_q = \prod_{i=1}^{+\infty} \left( 1 - \frac{T}{Tq^i} \right)^{-1} \in \mathbb{F}_q((1/T)) \]

**Question:** is the number \( \pi_q \) transcendental over \( \mathbb{F}_q(T) \)?

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\[ f(z) = \prod_{i=1}^{+\infty} \left( 1 - \frac{T}{zq^i} \right)^{-1} \in \mathbb{F}_q(T)\{1/z\}. \]

We get:

\[ \pi_q = f(T), \]
\[ \pi_q = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{T q^i}\right)^{-1} \in \mathbb{F}_q((1/T)) \]

**Question**: is the number \( \pi_q \) transcendental over \( \mathbb{F}_q(T) \)?

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We get:

\[ \pi_q = f(T), \]

\[ f(z^q) = \left(1 - \frac{T}{z^q}\right) f(z). \]
\[ \pi_q = \prod_{i=1}^{+\infty} \left(1 - \frac{T}{Tq^i}\right)^{-1} \in \mathbb{F}_q((1/T)) \]

**Question:** is the number \( \pi_q \) transcendental over \( \overline{\mathbb{F}_q(T)} \)?

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We get:

\[ \pi_q = f(T), \]

\[ f(z^q) = \left(1 - \frac{T}{z^q}\right)f(z). \]
Theorem 4.1 (F.)

Let $n \geq 1$, $d \geq 2$ be two integers and $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$ be functions satisfying $d$-Mahler System (2). Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$, be a regular number with respect to System (2). Then

$$\text{trdeg}_{\mathbb{K}}\{f_1(\alpha), \ldots, f_n(\alpha)\} = \text{trdeg}_{\mathbb{K}(z)}\{f_1(z), \ldots, f_n(z)\}.$$
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Definition 4.2

We say that a number \( \alpha \in \mathbb{K} \) is **regular** with respect to System (2) if for all integer \( k \geq 0 \), the number \( \alpha^{d^k} \) is neither a pole of the matrix \( A(z) \) nor a pole of the matrix \( A^{-1}(z) \).
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When \( \mathbb{K} \) is a number field, this result is proved by Ku. Nishioka in 1991.
Limits:

1. We deal with algebraic independence of functions, which is an open question in general for Mahler functions. Linear independence of such functions is easier (Algorithm of B. Adamczewski and C. Faverjon in characteristic zero).
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2. If the Mahler function $f(z)$ is transcendental, we cannot conclude that $f(\alpha)$ is too.
Theorem 4.2 (F.)

We continue with the assumptions of Theorem 4.1.
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Then, for every polynomial \( P(X_1, \ldots, X_n) \in \overline{K}[X_1, \ldots, X_n] \) homogeneous in \( X_1, \ldots, X_n \) such that

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P(f_1(\alpha), \ldots, f_n(\alpha)) = 0,
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there exists a polynomial $Q(z, X_1, \ldots, X_n) \in \overline{\mathbb{K}}[z][X_1, \ldots, X_n]$ homogeneous in $X_1, \ldots, X_n$ such that

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\]

and

\[
Q(\alpha, X_1, \ldots, X_n) = P(X_1, \ldots, X_n).
\]
Corollary 4.1

We continue with the assumptions of Theorem 4.2. If the functions $f_1(z), \ldots, f_n(z)$ are linearly independent over $\overline{K}(z)$, then, the numbers $f_1(\alpha), \ldots, f_n(\alpha)$ are linearly independent over $\overline{K}$. 
Corollary 4.2

Let $f(z) \in K\{z\}$ be a $d$-Mahler transcendental function over $K(z)$. 
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Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ such that $\alpha$ is in the disc of convergence of $f(z)$.
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Then, we have the following.
Corollary 4.2

Let \( f(z) \in \mathbb{K}\{z\} \) be a \( d \)-Mahler transcendental function over \( \mathbb{K}(z) \).

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Let us assume that the extension \( \overline{\mathbb{K}}(z)(f(z))_{\sigma_d} \) is regular over \( \overline{\mathbb{K}}(z) \).

Then, we have the following.

1. The number \( f(\alpha) \) is either transcendental or in \( \overline{\mathbb{K}}(\alpha) \).
2. If \( \alpha \) is a regular number with respect to the minimal \( d \)-Mahler System satisfied by \( f(z) \), then \( f(\alpha) \) is transcendental over \( \overline{\mathbb{K}} \).
When $\mathbb{K}$ is a number field, the analogue of the theorem and corollaries are proved by B. Adamczewski and C. Faverjon, as a consequence of the work of P. Philippon (2017).
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**Cobham conjecture.** Let

$$\sqrt{2} = \sum_{n=0}^{+\infty} a_n 10^{-n}$$

be the decomposition of $\sqrt{2}$ in base 10.

Let us assume by *contradiction* that $(a_n)_n$ is automatic.

Then the series

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Then $f \left( \frac{1}{10} \right)$ is *transcendental or rational*: contradiction.
Definition 4.3

If $f_1(z), \ldots, f_n(z)$ are $d$-Mahler functions over $\mathbb{K}$, we say that the Mahler extension $\mathcal{E} = \overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\mathbb{K}}(z)$ if every algebraic element of $\mathcal{E}$ belongs to $\overline{\mathbb{K}}(z)$. 
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1. In characteristic zero, every Mahler extension is regular. Because a Mahler function is either transcendental or rational.

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Example:
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Example:

Let consider the $p$-Mahler system:

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\begin{pmatrix}
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But the function \( f_2(z) \) is **not rational**.

It follows that the extension \( \mathcal{E} = \overline{\mathbb{K}}(z)(f_1(z), f_2(z)) \) is **not regular** over \( \overline{\mathbb{K}}(z) \).
Let $\alpha \in \overline{K}$, $0 < |\alpha| < 1$ and $\lambda = f_2(\alpha) \in \overline{K}$. 
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Then, $\lambda f_1(\alpha) - f_2(\alpha) = 0$ is a non-trivial linear relation between $f_1(\alpha)$ and $f_2(\alpha)$ over $\overline{K}$. 
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However, there is **no non-trivial linear relation** between the function \( f_1(z) \) and \( f_2(z) \) over \( \overline{K}(z) \), because \( f_2(z) \) is not rational.
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However, there is no non-trivial linear relation between the function $f_1(z)$ and $f_2(z)$ over $\overline{K}(z)$, because $f_2(z)$ is not rational.

Hence, the conclusion of Theorem 4.2 does not hold in this case.
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3. Concrete examples usually are in the case $p \mid d$. 
We obtained results of **functional transcendence and algebraic independence** for inhomogeneous Mahler equations of **order** 1. By a generalisation of results of L. Denis.
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2. For general Mahler equations in characteristic zero: **Galois theory**. Work of J. Roques for Mahler equations of order 2.
Thank you for your attention!