Weakly and Strongly Aperiodic SFTs on Baumslag-Solitar Groups

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Joint work with Etienne Moutot

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SDA2
December 3rd 2020
\[ \mathcal{A} = \{ \begin{array}{c} \hspace{1cm} \end{array} \} \]

\[ \mathcal{X} = \{ x \in \mathcal{A}^{\mathbb{Z}^2} \mid \text{adjacent tiles of } \mathcal{A} \text{ match in } x \} \]

is a Subshift of Finite Type (SFT).
$\mathcal{A} = \{ \begin{array}{c} \text{tiles} \\ \end{array} \} $

$\mathcal{X} = \{ x \in \mathcal{A}^{\mathbb{Z}^2} \mid \text{adjacent tiles of } \mathcal{A} \text{ match in } x \} $

is a Subshift of Finite Type (SFT).
\[ \mathbb{Z}^2 = \langle a, b \mid bab^{-1} = a \rangle \]
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Vertices = group elements
Edges $g - ga$ and $g - gb$
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Cycles following the relation
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Vertices = group elements

Edges \( g \to ga \) and \( g \to gb \)

Cycles following the relation

\[ a \]
\[ b \]
\[ b^{-1} \]
\[ \mathbb{Z}^2 = \langle a, b \mid bab^{-1} = a \rangle \]

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\[ \mathbb{Z}^2 \text{ acts on } X \text{ by translation:} \]

\[ ((0, -1) \cdot x)(0, 0) = x(-(0, -1)) = x(0, 1) \]
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\[ \mathbb{Z}^2 \text{ acts on } X \text{ by translation:} \]

\[ ((m, n) \cdot x)(p, q) = x(-(m, n) + (p, q)) \]
\( \mathbb{Z}^2 \) acts on \( x \in X \) by translation:

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((m, n) \cdot x)(p, q) = x(- (m, n) + (p, q))
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SFT \( X \subset A^G \) on any group \( G \). Let \( x \in X \), we have the \textit{shift action}

\[
(g \cdot x)(h) = x(g^{-1}h).
\]
$\mathbb{Z}^2$ acts on $x \in X$ by translation:

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SFT $X \subset A^G$ on any group $G$. Let $x \in X$, we have the shift action

$$(g \cdot x)(h) = x(g^{-1}h).$$

For $x \in X$, $Stab_G(x) = \{g \in G \mid g \cdot x = x\}$. 
$\mathbb{Z}^2$ acts on $x \in X$ by translation:

$$((m, n) \cdot x)(p, q) = x(-(m, n) + (p, q))$$

SFT $X \subset \mathcal{A}^G$ on any group $G$. Let $x \in X$, we have the *shift action*

$$(g \cdot x)(h) = x(g^{-1}h).$$

For $x \in X$, $Stab_G(x) = \{ g \in G \mid g \cdot x = x \}$.
For $x \in X$, $Orb_G(x) = \{ y \in X \mid \exists g \in G, g \cdot x = y \}$. 
Definition

Let $X$ be a nonempty SFT on a group $G$. 

IMPORTANT: we require ALL configurations of $X$ to look like that!!
Definition

Let $X$ be a nonempty SFT on a group $G$. $X$ is a \textit{weakly aperiodic} SFT if $\forall x \in X, \left|\text{Orb}_G(x)\right| = +\infty$.

IMPORTANT: we require ALL configurations of $X$ to look like that!!
**Definition**

Let $X$ be a nonempty SFT on a group $G$.

- $X$ is a **weakly aperiodic** SFT if $\forall x \in X, |Orb_G(x)| = +\infty$.
- $X$ is a **strongly aperiodic** SFT if $\forall x \in X, Stab_G(x) = \{e\}$.

**IMPORTANT:** we require ALL configurations of $X$ to look like that!!
Theorem (Berger 66, Robinson 71...) 

There exists a strongly aperiodic SFT on $\mathbb{Z}^2$.

Tiles can be rotated and reflected.
Theorem (Berger 66, Robinson 71...)

There exists a strongly aperiodic SFT on $\mathbb{Z}^2$. 
Remark

If $G$ is infinite, strongly aperiodic $\Rightarrow$ weakly aperiodic.
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Theorem (folklore)

A nonempty SFT $X$ on $\mathbb{Z}^2$ is weakly aperiodic if and only if it is strongly aperiodic.
Remark
If $G$ is infinite, strongly aperiodic $\Rightarrow$ weakly aperiodic.

Theorem (folklore)
A nonempty SFT $X$ on $\mathbb{Z}^2$ is weakly aperiodic if and only if it is strongly aperiodic.

Question
To what extent can these complex configurations emerge from local rules? Does it depend on the group structure?
$BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle, m, n \in \mathbb{Z}$
$BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle, m, n \in \mathbb{Z}$

$BS(1, 1) = \mathbb{Z}^2$
BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle, \ m, n \in \mathbb{Z}

BS(1, 1) = \mathbb{Z}^2

Example: BS(1, 2)
$BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle, m, n \in \mathbb{Z}$

$BS(1, 1) = \mathbb{Z}^2$

Example: $BS(1, 2)$
BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle, m, n \in \mathbb{Z}

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Example: $BS(1, 2)$
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BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle, \ m, n \in \mathbb{Z}
\]

\[
BS(1, 1) = \mathbb{Z}^2
\]

Example: \( BS(1, 2) \)
Motivation

Results on BS Groups

Weak-but-not-strong aperiodicity for BS(1,n)

Conclusion

\[
BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle, \ m, n \in \mathbb{Z}
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Example: \(BS(1, 2)\)
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Figure: Wikipedia, page *Baumslag-Solitar Group*

**Figure:** Wikipedia, page *Baumslag-Solitar Group*
Another example, \( BS(2, 3) \):

\[
\begin{align*}
g.b & \quad g.ba & \quad g.ba^2 \\
g & \quad g.a & \quad g.a^2 & \quad g.a^3
\end{align*}
\]
<table>
<thead>
<tr>
<th>Shape</th>
<th>Group</th>
<th>Strongly aperiodic SFT</th>
<th>Weakly-not-strongly aperiodic SFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>$\mathbb{Z}^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y+</td>
<td>$BS(1, n)$</td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
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<td>----------------------------------</td>
</tr>
<tr>
<td>[+]</td>
<td>[\mathbb{Z}^2]</td>
<td></td>
<td>(Folklore)</td>
</tr>
<tr>
<td>[Y] +</td>
<td>[BS(1, n)]</td>
<td></td>
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</tr>
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### Aperiodic SFTs on BS Groups

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<tbody>
<tr>
<td><img src="image1" alt="Shape 1" /></td>
<td>$\mathbb{Z}^2$</td>
<td>(Robinson)</td>
<td>(Folklore)</td>
</tr>
<tr>
<td><img src="image2" alt="Shape 2" /></td>
<td>$BS(1, n)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><img src="image3" alt="Shape 3" /></td>
<td>$BS(n, n)$</td>
<td></td>
<td>(Aubrun-Kari)</td>
</tr>
<tr>
<td><img src="image4" alt="Shape 4" /></td>
<td>$BS(m, n)$</td>
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<tr>
<td>$\Upsilon$+</td>
<td>$BS(1, n)$</td>
<td>Adapted from Aubrun-Kari</td>
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<tr>
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### Results on BS Groups

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<td>$\square$</td>
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<tr>
<td>X</td>
<td>$BS(n, n)$</td>
<td>Group theory and theorem by Jeandel</td>
<td>(Aubrun-Kari)</td>
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### Motivation

#### Results on BS Groups

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### Conclusion

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\[ \sigma_0 : \begin{cases} 
0 & \mapsto 0^{n-1}1 \\
1 & \mapsto 0^n 
\end{cases} \]
σ_0: \[
\begin{align*}
0 &\mapsto 0^{n-1}1 \\
1 &\mapsto 0^n
\end{align*}
\]

σ_r: \[
\begin{align*}
0 &\mapsto 0^{n-r-1}10^r \\
1 &\mapsto 0^n
\end{align*}
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\[ \sigma_0 : \begin{cases} 
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\[ \sigma_r : \begin{cases} 
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Weak-but-not-strong aperiodicity for BS(1,n)

σ₀ : \[
\begin{align*}
0 &\mapsto 0^{n-1}1 \\
1 &\mapsto 0^n
\end{align*}
\]

σᵣ : \[
\begin{align*}
0 &\mapsto 0^{n-r-1}10^r \\
1 &\mapsto 0^n
\end{align*}
\]

\(c \in \{0, 1\}\)

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0

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Aperiodic SFTs on BS Groups

Julien Esnay (IMT)

SDA2, 03/12/20
Theorem (E.-Moutot)

The resulting SFT $X_\sigma$ is weakly aperiodic.
**Theorem (E.-Moutot)**

The resulting SFT $X_\sigma$ is weakly aperiodic.

If some $a^k$ period, $k \in \mathbb{Z}$, then base word $u$ is $k$-periodic.
Theorem (E.-Moutot)

The resulting SFT $X_\sigma$ is weakly aperiodic.

If some $a^k$ period, $k \in \mathbb{Z}$, then base word $u$ is $k$-periodic.
It is also $nk$-periodic.
Theorem (E.-Moutot)
The resulting SFT $X_\sigma$ is weakly aperiodic.

If some $a^k$ period, $k \in \mathbb{Z}$, then base word $u$ is $k$-periodic.
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Then the words above $u$ are $k$-periodic.
Theorem (E.-Moutot)

The resulting SFT $X_\sigma$ is weakly aperiodic.

If some $a^k$ period, $k \in \mathbb{Z}$, then base word $u$ is $k$-periodic. It is also $nk$-periodic. Then the words above $u$ are $k$-periodic. By the pigeonhole principle, two of these words are identical.
Theorem (E.-Moutot)

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If some $a^k$ period, $k \in \mathbb{Z}$, then base word $u$ is $k$-periodic.
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By the pigeonhole principle, two of these words are identical.

Proposition

A fixpoint of any $\sigma_{i_k} \circ \cdots \circ \sigma_{i_1}$ is aperiodic.
Theorem (E.-Moutot)
The resulting SFT $X_\sigma$ is weakly aperiodic.

If some $a^k$ period, $k \in \mathbb{Z}$, then base word $u$ is $k$-periodic. It is also $nk$-periodic. Then the words above $u$ are $k$-periodic. By the pigeonhole principle, two of these words are identical.

Proposition
A fixpoint of any $\sigma_{i_k} \circ \cdots \circ \sigma_{i_1}$ is aperiodic.

No $a^k$ period for any configuration $\implies$ weak aperiodicity.
Theorem (E.-Moutot)

$X_\sigma$ contains some $b$-periodic configuration.
Theorem (E.-Moutot)

$X_\sigma$ contains some $b$-periodic configuration.

\[ \sigma_1 : \begin{cases} 0 &\mapsto 0^{n-2}10 \\ 1 &\mapsto 0^n \end{cases} \]

\[ w := \sigma_1(w) \]
Theorem (E.-Moutot)

$X_\sigma$ contains some $b$-periodic configuration.

\[ \sigma_1 : \begin{cases} 
0 \mapsto 0^{n-2}10 \\
1 \mapsto 0^n 
\end{cases} \]

\[ w := \sigma_1(w) \]
Theorem (E.-Moutot, 2020):

- Strongly aperiodic SFT on $BS(1, n)$;
- **Weakly-not-strongly aperiodic SFT on $BS(1, n)$**;
- Strongly aperiodic SFT on $BS(n, n)$.

Thank you for your attention!
Strongly aperiodic SFT on $BS(n, n)$:

$$BS(n, n) \triangleright H \cong \mathbb{Z} \times \mathbb{F}_n$$

with $H$ of finite index.
Strongly aperiodic SFT on $BS(n, n)$:

$$BS(n, n) \triangleright H \cong \mathbb{Z} \times \mathbb{F}_n$$

with $H$ of finite index.

**Theorem (Jeandel 2015)**

There exists a strongly aperiodic SFT on $\mathbb{Z} \times \mathbb{F}_n$. 
Strongly aperiodic SFT on $BS(n, n)$:

$$BS(n, n) \triangleright H \cong \mathbb{Z} \times F_n$$

with $H$ of finite index.

**Theorem (Jeandel 2015)**

There exists a strongly aperiodic SFT on $\mathbb{Z} \times F_n$.

**Theorem (Carroll-Penland 2015)**

A finitely generated group and a subgroup of finite index have the same aperiodicity.
Strongly aperiodic SFT on $BS(1, n)$:

$$f : \left[\frac{1}{3}, 2\right] \rightarrow \left[\frac{1}{3}, 2\right]$$

$$f(x) = \begin{cases} 
  f_1(x) = 2x & \text{if } x \in \left[\frac{1}{3}, 1\right) \\
  f_2(x) = \frac{1}{3}x & \text{if } x \in [1, 2] 
\end{cases}$$

There is no $k$ such that $f^k(x) = x$. 
Strongly aperiodic SFT on $BS(1, n)$:

$$f : [\frac{1}{3}, 2] \rightarrow [\frac{1}{3}, 2]$$

$$f(x) = \begin{cases} 
  f_1(x) = 2x & \text{if } x \in [\frac{1}{3}, 1) \\
  f_2(x) = \frac{1}{3}x & \text{if } x \in [1, 2]
\end{cases}$$

There is no $k$ such that $f^k(x) = x$. 

Diagram showing the action of $f$ and $f^2$ on the interval.
\[ f^{-2}(x) \quad f^{-2}(x) \quad f^{-2}(x) \quad f^{-2}(x) \]

\[ f^{-1}(x) \quad f^{-1}(x) \quad f^{-1}(x) \quad f^{-1}(x) \]

\[ x \]

No period following a: same reason as substitutions.

No period following b: no periodic orbit of f.

No combination of a and b: nice tree-like structure of BS(1, n).
No period following $a$: same reason as substitutions.
No period following $a$: same reason as substitutions.
No period following $b$: no periodic orbit of $f$. 
No period following $a$: same reason as substitutions.
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No combination of $a$ and $b$: nice tree-like structure of $BS(1, n)$. 