Weakly and Strongly Aperiodic SFTs on Baumslag-Solitar Groups

Julien Esnay
Joint work with Etienne Moutot

Institut de Mathématiques de Toulouse

SDA2
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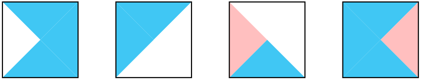
$\mathcal{A} = \{$  $\}$

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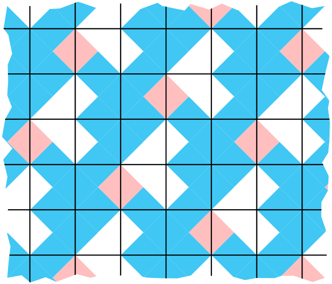
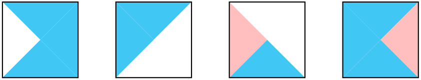
is a Subshift of Finite Type (SFT).



$\mathcal{A} = \{$  $\}$

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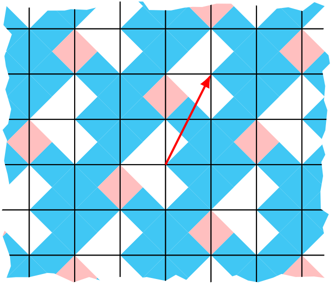

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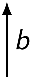

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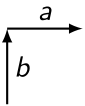

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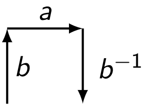

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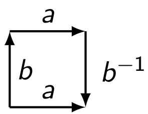

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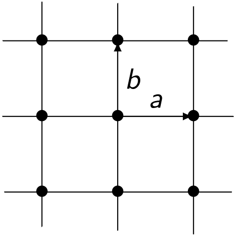

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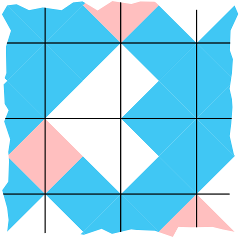

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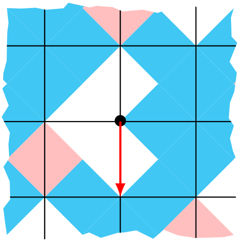

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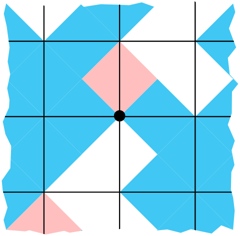

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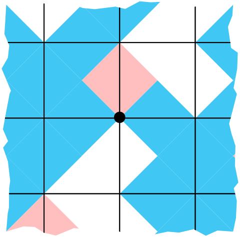

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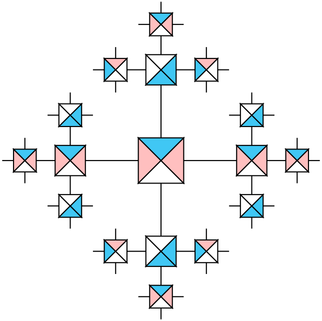

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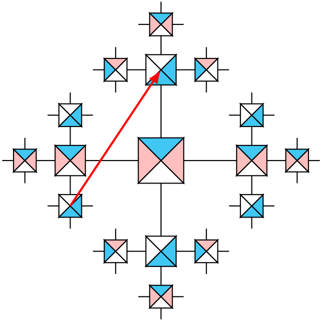

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For $x \in X$, $Orb_G(x) = \{y \in X \mid \exists g \in G, g \cdot x = y\}$.

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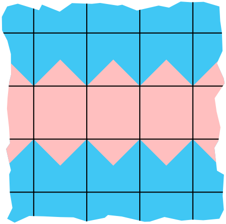

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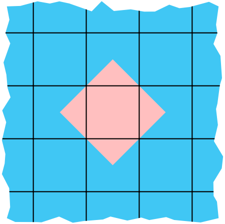

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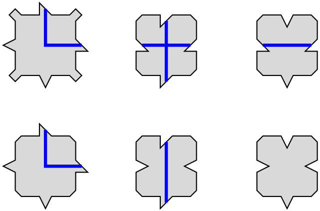
Theorem (Berger 66, Robinson 71...)

There exists a strongly aperiodic SFT on $\mathbb{Z}^2$.

Tiles can be rotated and reflected.



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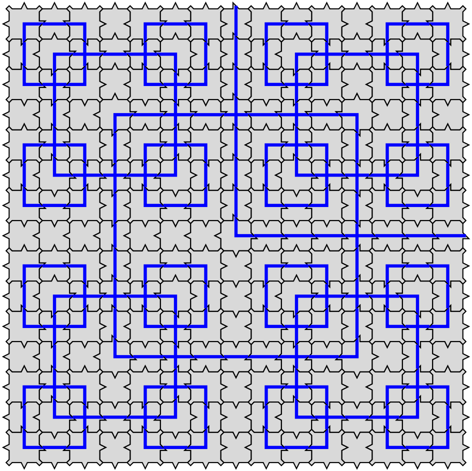

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Question

To what extent can these complex configurations emerge from local rules? Does it depend on the group structure?

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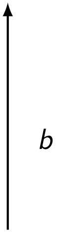

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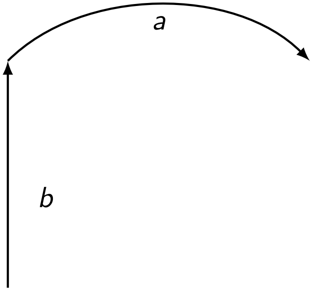

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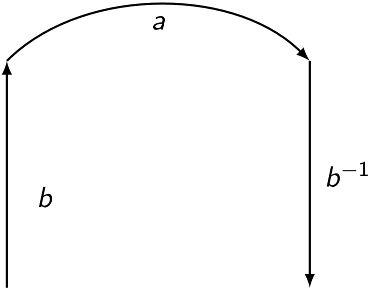

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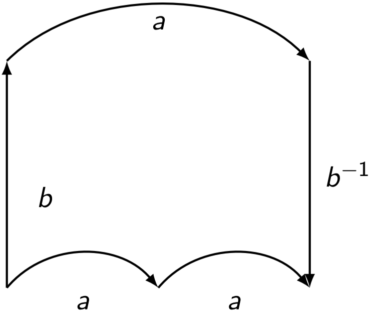

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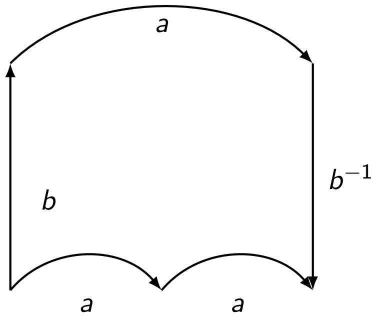
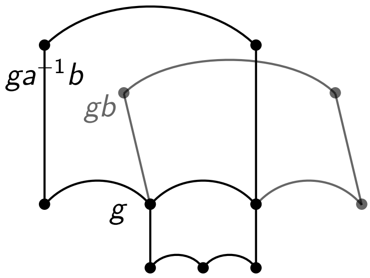
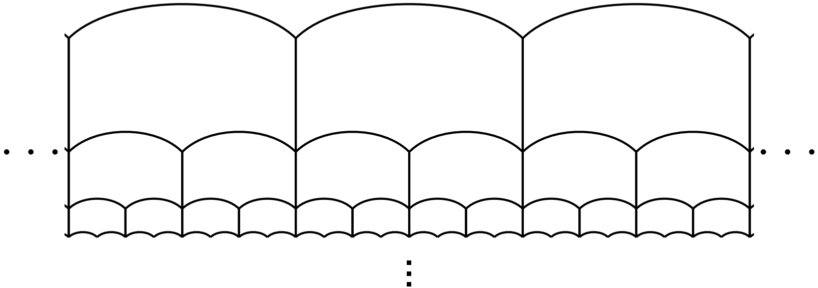
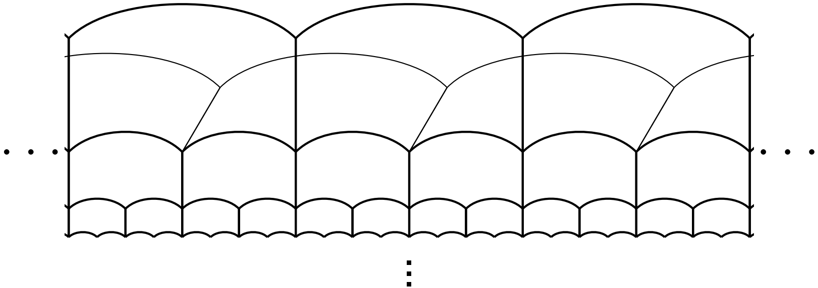
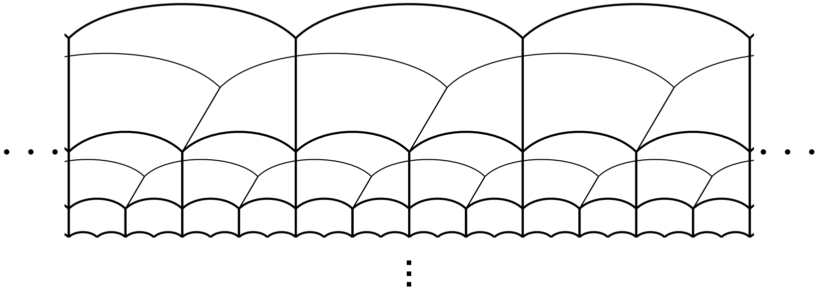
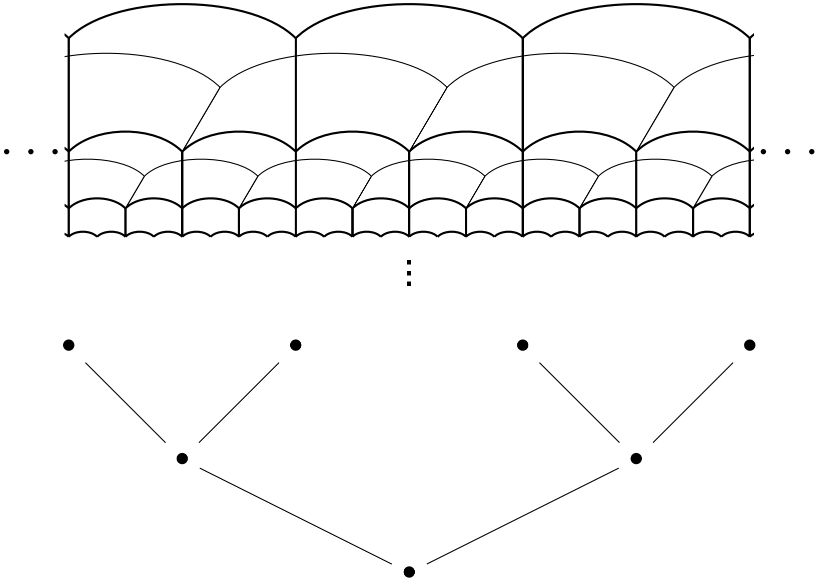












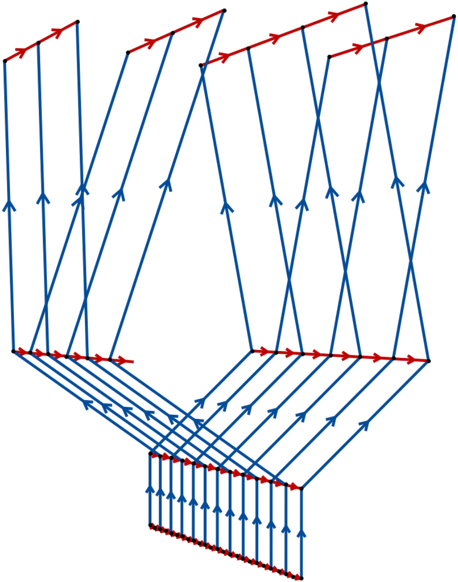
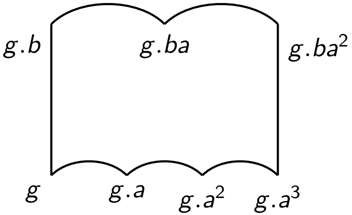
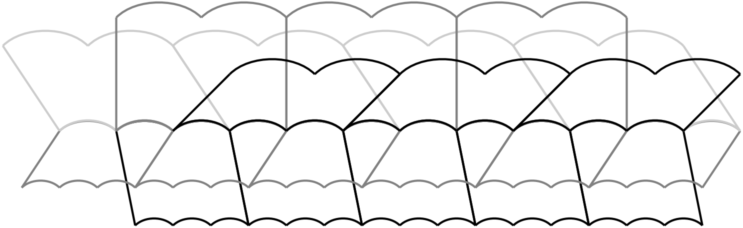


Figure: Wikipedia, page *Baumslag-Solitar Group*

Another example, $BS(2,3)$:





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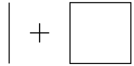
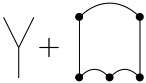
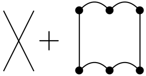
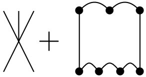
	$\mathbb{Z}^2$		
	$BS(1, n)$		
	$BS(n, n)$		
	$BS(m, n)$		

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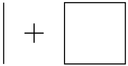
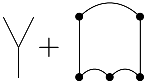
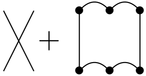
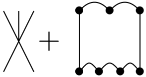
	$\mathbb{Z}^2$		(Folklore)
	$BS(1, n)$		
	$BS(n, n)$		
	$BS(m, n)$		

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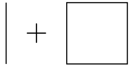
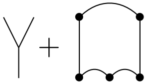
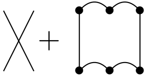
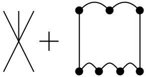
	$\mathbb{Z}^2$	(Robinson)	(Folklore)
	$BS(1, n)$		
	$BS(n, n)$		
	$BS(m, n)$		

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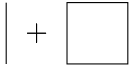
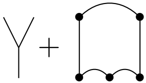
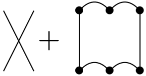
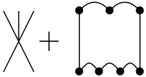
	$\mathbb{Z}^2$	(Robinson)	(Folklore)
	$BS(1, n)$		
	$BS(n, n)$		(Aubrun-Kari)
	$BS(m, n)$		(Aubrun-Kari)

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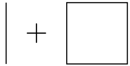
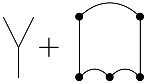
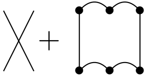
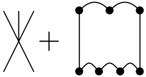
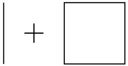
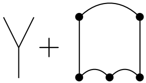
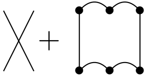
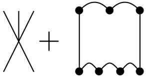
	$\mathbb{Z}^2$	(Robinson)	(Folklore)
	$BS(1, n)$	Adapted from Aubrun-Kari	
	$BS(n, n)$		(Aubrun-Kari)
	$BS(m, n)$		(Aubrun-Kari)

Shape	Group	Strongly aperiodic SFT	Weakly-not-strongly aperiodic SFT
	$\mathbb{Z}^2$	(Robinson)	(Folklore)
	$BS(1, n)$	Adapted from Aubrun-Kari	Using substitutions
	$BS(n, n)$		(Aubrun-Kari)
	$BS(m, n)$		(Aubrun-Kari)

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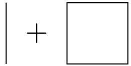
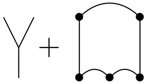
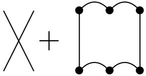
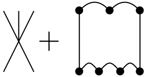
	$\mathbb{Z}^2$	(Robinson)	(Folklore)
	$BS(1,n)$	Adapted from Aubrun-Kari	Using substitutions
	$BS(n,n)$	Group theory and theorem by Jeandel	(Aubrun-Kari)
	$BS(m,n)$		(Aubrun-Kari)

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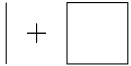
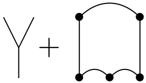
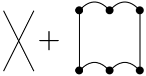
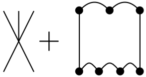
	$\mathbb{Z}^2$	(Robinson)	(Folklore)
	$BS(1,n)$	Adapted from Aubrun-Kari	Using substitutions
	$BS(n,n)$	Group theory and theorem by Jeandel	(Aubrun-Kari)
	$BS(m,n)$	?	(Aubrun-Kari)

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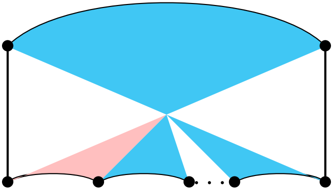

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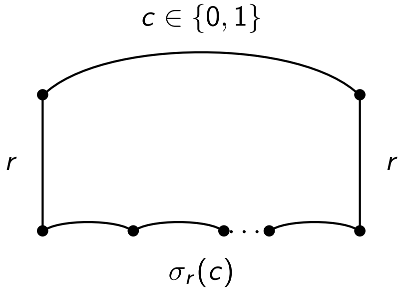

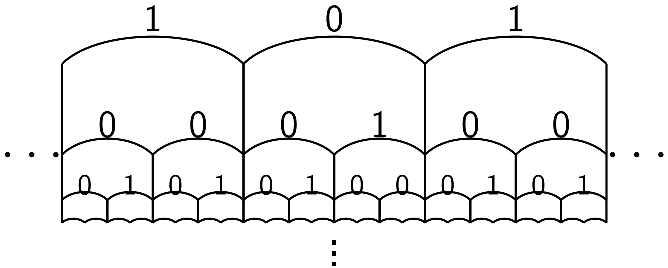
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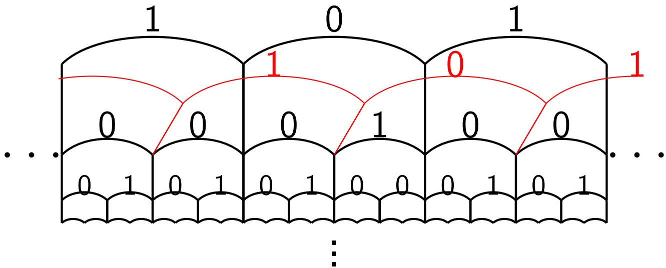
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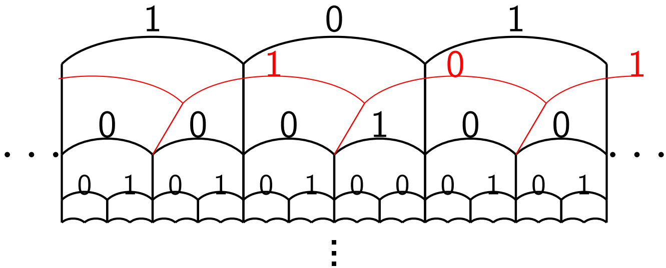
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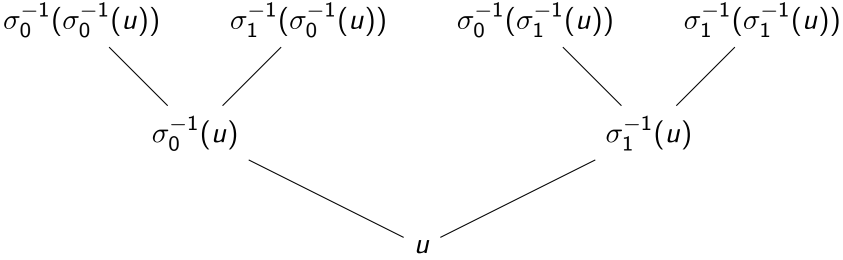
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No a^k period for any configuration $\Rightarrow$ weak aperiodicity.

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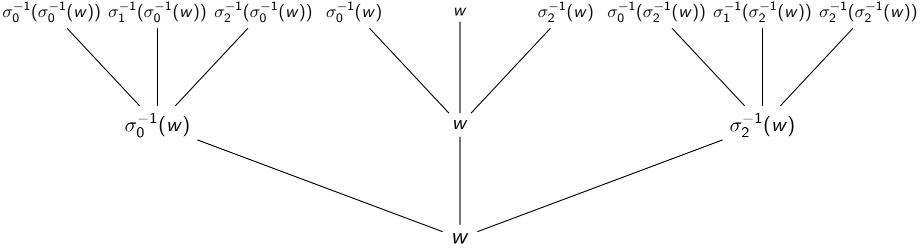

Theorem (E.-Moutot, 2020):

- Strongly aperiodic SFT on $BS(1, n)$;
- **Weakly-not-strongly aperiodic SFT on** $BS(1, n)$;
- Strongly aperiodic SFT on $BS(n, n)$.

Thank you for your attention!

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$$BS(n,n) \rhd H \cong \mathbb{Z} \times \mathbb{F}_n$$

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A finitely generated group and a subgroup of finite index have the same aperiodicity.

Strongly aperiodic SFT on $BS(1,n)$:

$$f\colon [\tfrac{1}{3},2] \to [\tfrac{1}{3},2]$$
$$f(x) = \begin{cases} f_1(x) = 2x & \text{if } x \in [\frac{1}{3},1) \\ f_2(x) = \frac{1}{3}x & \text{if } x \in [1,2] \end{cases}$$

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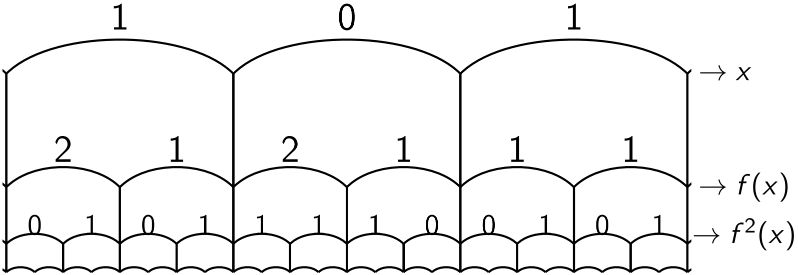

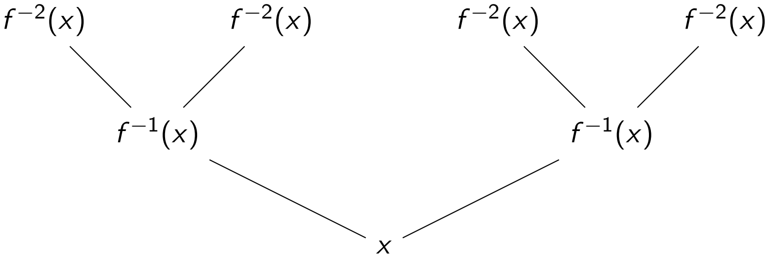
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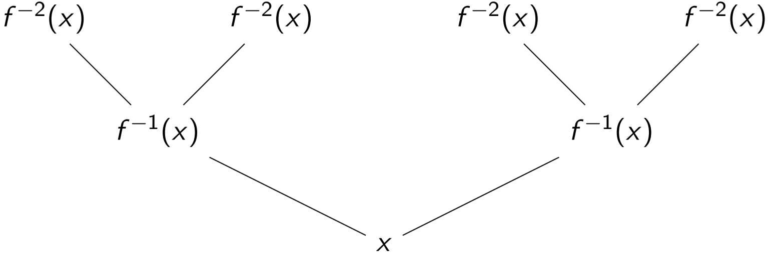

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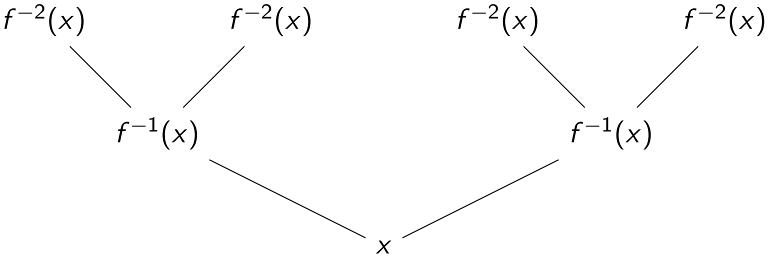

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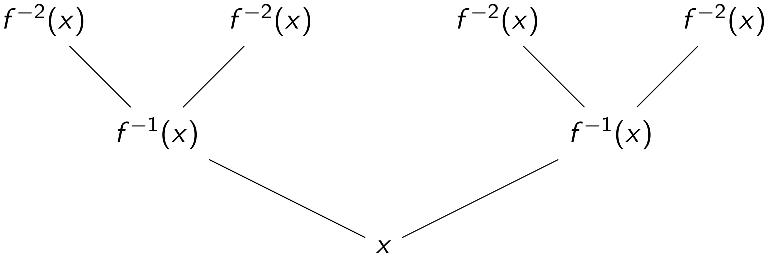

No period following a: same reason as substitutions.
No period following b: no periodic orbit of f.
No combination of a and b: nice tree-like structure of $BS(1, n)$.